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Introduction to Irrational Numbers Using Geometry

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[1] Each lesson is ONE day, and ONE day is considered a 45-minute period.

Grade 8 • Module 7

Introduction to Irrational Numbers Using Geometry

OVERVIEW

The module begins with work related to the Pythagorean theorem and right triangles. Before the lessons of this module are presented to students, it is important that the lessons in Modules 2 and 3 related to the Pythagorean theorem are taught (M2: Lessons 15 and 16, M3: Lessons 13 and 14). In Modules 2 and 3, students used the Pythagorean theorem to determine the unknown side length of a right triangle. In cases where the side length was an integer, students computed the length. When the side length was not an integer, students left the answer in the form of $x^2 = c$, where c was not a perfect square number. Those solutions are revisited and are the motivation for learning about square roots and irrational numbers in general.

In Topic A, students learn the notation related to roots (**8.EE.A.2**). The definition for irrational numbers relies on students' understanding of rational numbers; that is, students know that rational numbers are points on a number line (**6.NS.C.6**) and that every quotient of integers (with a nonzero divisor) is a rational number (**7.NS.A.2**). Then, irrational numbers are numbers that can be placed in their approximate positions on a number line and not expressed as a quotient of integers. Though the term *irrational* is not introduced until Topic B, students learn that irrational numbers exist and are different from rational numbers. Students learn to find positive square roots and cube roots of expressions and know that there is only one such number (**8.EE.A.2**). Topic A includes some extension work on simplifying perfect square factors of radicals in preparation for Algebra I.

In Topic B, students learn that to get the decimal expansion of a number (**8.NS.A.1**), they must develop a deeper understanding of the long division algorithm learned in Grades 6 and 7 (**6.NS.B.2**, **7.NS.A.2d**). Students show that the decimal expansion for rational numbers repeats eventually, in some cases with zeros, and they can convert the decimal form of a number into a fraction (**8.NS.A.2**). Students learn a procedure to get the approximate decimal expansion of numbers like $\sqrt{2}$ and $\sqrt{5}$ and compare the size of these irrational numbers using their rational approximations. At this point, students learn that the definition of an irrational number is a number that is not equal to a rational number (**8.NS.A.1**). In the past, irrational numbers may have been described as numbers that are infinite decimals that cannot be expressed as a fraction, like the number π. This may have led to confusion about irrational numbers because until now, students did not know how to write repeating decimals as fractions; additionally, students frequently approximated π using $\frac{22}{7}$, which led to more confusion about the definition of irrational numbers. Defining irrational numbers as those that are not equal to rational numbers provides an important guidepost for students' knowledge of numbers. Students learn that an irrational number is something quite different from other numbers they have studied before. They are infinite decimals that can only be expressed by a decimal approximation. Now that students know that irrational numbers can be approximated, they extend their knowledge of the number line gained in Grade 6 (**6.NS.C.6**) to include being able to position irrational numbers on a line diagram in their approximate locations (**8.NS.A.2**).

Topic C revisits the Pythagorean theorem and its applications in a context that now includes the use of square roots and irrational numbers. Students learn another proof of the Pythagorean theorem involving areas of squares off of each side of a right triangle (**8.G.B.6**). Another proof of the converse of the Pythagorean theorem is presented, which requires an understanding of congruent triangles (**8.G.B.6**). With the concept of square roots firmly in place, students apply the Pythagorean theorem to solve real-world and mathematical problems to determine an unknown side length of a right triangle and the distance between two points on the coordinate plane (**8.G.B.7**, **8.G.B.8**).

In Topic D, students learn that radical expressions naturally arise in geometry, such as the height of an isosceles triangle or the lateral length of a cone. The Pythagorean theorem is applied to three-dimensional figures in Topic D as students learn some geometric applications of radicals and roots (**8.G.B.7**). In order for students to determine the volume of a cone or sphere, they must first apply the Pythagorean theorem to determine the height of the cone, or the radius of the sphere. Students learn that truncated cones are solids obtained by removing the top portion above a plane parallel to the base. Students know that to find the volume of a truncated cone they must access and apply their knowledge of similar figures learned in Module 3. Their work with truncated cones is an exploration of solids that is not formally assessed. In general, students solve real-world and mathematical problems in three dimensions in Topic D (**8.G.C.9**). For example, now that students can compute with cube roots and understand the concept of rate of change, students compute the average rate of change in the height of the water level when water is poured into a conical container at a constant rate. Students also use what they learned about the volume of cylinders, cones, and spheres to compare volumes of composite solids.

It is recommended that students have access to a calculator to complete the End-of-Module Assessment but that they complete the Mid-Module Assessment without one.

The discussion of infinite decimals and the conversion of fractions to decimals in this module is taken from the following source:

H. Wu, *Mathematics of the Secondary School Curriculum, Volume III* (to appear in 2015).

Focus Standards

Know that there are numbers that are not rational, and approximate them by rational numbers.

8.NS.A.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.

8.NS.A.2 Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of* $\sqrt{2}$*, show that* $\sqrt{2}$ *is between* 1 *and* 2*, then between* 1.4 *and* 1.5*, and explain how to continue on to get a better approximation.*

Work with radicals and integer exponents.[2]

8.EE.A.2 Use square root and cube root symbols to represent solutions to the equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

Understand and apply the Pythagorean Theorem.

8.G.B.6 Explain a proof of the Pythagorean Theorem and its converse.

8.G.B.7 Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.

8.G.B.8 Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.

Solve real-world and mathematical problems involving volume of cylinders, cones, and spheres.

8.G.C.9 Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.[3]

Foundational Standards

Compute fluently with multi-digit numbers and find common factors and multiples.

6.NS.B.2 Fluently divide multi-digit numbers using the standard algorithm.

Apply and extend previous understandings of numbers to the system of rational numbers.

6.NS.C.6 Understand a rational number as a point on the number line. Extend number line diagrams and coordinate axes familiar from previous grades to represent points on the line and in the plane with negative number coordinates.

a. Recognize opposite signs of numbers as indicating locations on opposite sides of 0 on the number line; recognize that the opposite of the opposite of a number is the number itself, e.g., $-(-3) = 3$ and that 0 is its own opposite.

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[2] The balance of this cluster is taught in Module 1.

[3] Solutions that introduce irrational numbers are allowed in this module.

b. Understand signs of numbers in ordered pairs as indicating locations in quadrants of the coordinate plane; recognize that when two ordered pairs differ only by signs, the locations of the points are related by reflections across one or both axes.

c. Find and position integers and other rational numbers on a horizontal and vertical number line diagram; find and position pairs of integers and other rational numbers on a coordinate plane.

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

7.NS.A.2 Apply and extend previous understandings of multiplication and division of fractions to multiply and divide rational numbers.

a. Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as $(-1)(-1) = 1$ and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world contexts.

b. Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-\frac{p}{q} = \frac{-p}{q} = \frac{p}{-q}$. Interpret quotients of rational numbers by describing real-world contexts.

c. Apply properties of operations as strategies to multiply and divide rational numbers.

d. Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats.

Draw, construct, and describe geometrical figures and describe the relationships between them.

7.G.A.2 Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.

7.G.B.6 Solve real-world and mathematical problems involving area, volume, and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Focus Standards for Mathematical Practice

MP.6 **Attend to precision**. Students begin attending to precision by recognizing and identifying numbers as rational or irrational. Students know the definition of an irrational number and can represent the number in different ways, e.g., as a root, as a non-repeating decimal block, or as a symbol such as π. Students will attend to precision when clarifying the difference between an exact value of an irrational number compared to the decimal approximation of the irrational number. Students use appropriate symbols and definitions when they work through proofs of the Pythagorean theorem and its converse. Students know and apply formulas related to volume of cones and truncated cones.

MP.7 **Look for and make use of structure**. Students learn that a radicand can be rewritten as a product and that sometimes one or more of the factors of the product can be simplified to a rational number. Students look for structure in repeating decimals, recognize repeating blocks, and know that every fraction is equal to a repeating decimal. Additionally, students learn to see composite solids as made up of simpler solids. Students interpret numerical expressions as representations of volumes of complex figures.

MP.8 **Look for and express regularity in repeated reasoning**. While using the long division algorithm to convert fractions to decimals, students recognize that when a sequence of remainders repeats, the decimal form of the number will contain a repeat block. Students recognize that when the decimal expansion of a number does not repeat or terminate, the number is irrational and can be represented with a method of rational approximation using a sequence of rational numbers to get closer and closer to the given number.

Terminology

New or Recently Introduced Terms

- **Chord** (Let P and Q be two points on the circle. Then, PQ is called a *chord* of the circle.)
- **Cube Root** (The *cube root* of a number b is equal to a if $a^3 = b$. It is denoted by $\sqrt[3]{b}$.)
- **Infinite Decimals** (*Infinite decimals* are decimals that neither repeat nor terminate.)
- **Irrational Numbers** (*Irrational numbers* are numbers that are not rational.)
- **Perfect Square** (A *perfect square* is the square of an integer.)
- **Rational Approximation** (*Rational approximation* is the method for determining the approximated rational form of an irrational number.)
- **A Square Root of a Number** (*A square root of a number* x is a number whose square is x. In symbols, a square root of x is a number a such that $a^2 = x$. Negative numbers do not have any square roots, zero has exactly one square root, and positive numbers have two square roots.)
- **The Square Root of a Number** (Every positive real number x has a unique positive square root called *the square root of a number* or *principle square root* of x; it is denoted $\sqrt{x}$. The square root of zero is zero.)
- **Truncated Cone** (A *truncated cone* is a solid obtained from a cone by removing the top portion above a plane parallel to the base.)

Familiar Terms and Symbols[4]

- Decimal Expansion
- Finite Decimals
- Number Line
- Rate of Change
- Rational Number
- Volume

Suggested Tools and Representations

- 3D models (truncated cone, pyramid)
- Scientific Calculator

Rapid White Board Exchanges

Implementing an RWBE requires that each student be provided with a personal white board, a white board marker, and an eraser. An economic choice for these materials is to place two sheets of tag board (recommended) or cardstock, one red and one white, into a sheet protector. The white side is the "paper" side that students write on. The red side is the "signal" side, which can be used for students to indicate they have finished working—"Show red when ready." Sheets of felt cut into small squares can be used as erasers.

An RWBE consists of a sequence of 10 to 20 problems on a specific topic or skill that starts out with a relatively simple problem and progressively gets more difficult. The teacher should prepare the problems in a way that allows him or her to reveal them to the class one at a time. A flip chart or PowerPoint presentation can be used, or the teacher can write the problems on the board and either cover some with paper or simply write only one problem on the board at a time.

The teacher reveals, and possibly reads aloud, the first problem in the list and announces, "Go." Students work the problem on their personal white boards as quickly as possible. Depending on teacher preference, students can be directed to hold their work up for their teacher to see their answers as soon as they have the answer ready or to turn their white board face down to show the red side when they have finished. In the latter case, the teacher says, "Hold up your work," once all students have finished. The teacher gives immediate feedback to each student, pointing and/or making eye contact with the student and responding with an affirmation for correct work, such as "Good job!", "Yes!", or "Correct!", or responding with guidance for incorrect work such as "Look again," "Try again," "Check your work," etc. Feedback can also be more specific, such as "Watch your division facts," or "Error in your calculation."

If many students have struggled to get the answer correct, go through the solution of that problem as a class before moving on to the next problem in the sequence. Fluency in the skill has been established when the class is able to go through a sequence of problems leading up to and including the level of the relevant student objective, without pausing to go through the solution of each problem individually.

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[4] These are terms and symbols students have seen previously.

Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	8.NS.A.1, 8.NS.A.2, 8.EE.A.2
End-of-Module Assessment Task	After Topic D	Constructed response with rubric	8.G.B.6, 8.G.B.7, 8.G.B.8, 8.G.C.9

Topic A:

Square and Cube Roots

8.NS.A.1, 8.NS.A.2, 8.EE.A.2

Focus Standards:	8.NS.A.1	Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.
	8.NS.A.2	Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of* $\sqrt{2}$, *show that* $\sqrt{2}$ *is between* 1 *and* 2, *then between* 1.4 *and* 1.5, *and explain how to continue to get better approximations.*
	8.EE.A.2	Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.
Instructional Days:	5	
Lesson 1:	The Pythagorean Theorem (P)[1]	
Lesson 2:	Square Roots (S)	
Lesson 3:	Existence and Uniqueness of Square Roots and Cube Roots (S)	
Lesson 4:	Simplifying Square Roots (Optional) (P)	
Lesson 5:	Solving Radical Equations (P)	

The use of the Pythagorean theorem to determine side lengths of right triangles motivates the need for students to learn about square roots and irrational numbers in general. While students have previously applied the Pythagorean theorem using perfect squares, students begin by estimating the length of an unknown side of a right triangle in Lesson 1 by determining which two perfect squares a squared number is between. This leads them to know between which two positive integers the length must be. In Lesson 2, students are introduced to the notation and meaning of square roots. The term and formal definition for

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[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

irrational numbers is not given until Topic B, but students know that many of this type of number exist between the positive integers on the number line. That fact allows students to place square roots on a number line in their approximate position using perfect square numbers as reference points. In Lesson 3, students are given proof that the square or cube root of a number exists and is unique. Students then solve simple equations that require them to find the square root or cube root of a number. These will be in the form $x^2 = p$ or $x^3 = p$, where p is a positive rational number. In the optional Lesson 4, students learn that a square root of a number can be expressed as a product of its factors and use that fact to simplify the perfect square factors. For example, students know that they can rewrite $\sqrt{18}$ as $\sqrt{3^2 \times 2} = \sqrt{3^2} \times \sqrt{2} = 3 \times \sqrt{2} = 3\sqrt{2}$. The work in this lesson prepares students for what they may need to know in Algebra I to simplify radicals related to the quadratic formula. Some solutions in subsequent lessons are in simplified form, but these may be disregarded if Lesson 4 is not used. In Lesson 5, students solve multi-step equations that require students to use the properties of equality to transform an equation until it is in the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number.

Lesson 1: The Pythagorean Theorem

Student Outcomes

- Students know that they can estimate the length of a side of a right triangle as a number between two integers and identify the integer to which the length is closest.

Lesson Notes

Before beginning this lesson, it is imperative that students are familiar with the lessons in Modules 2 and 3 that relate to the Pythagorean theorem. This lesson assumes knowledge of the theorem and its basic applications. Students should not use calculators during this lesson.

In this lesson, students are exposed to expressions that involve irrational numbers, but they will not learn the definition of an irrational number until Topic B. It is important for students to understand that these irrational numbers can be approximated, but it is not yet necessary that they know the definition.

Scaffolding:

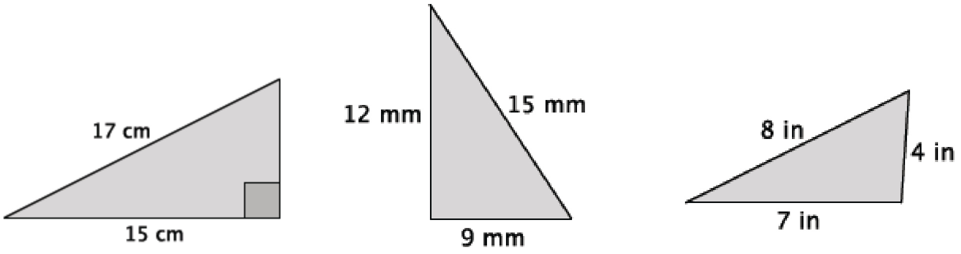
In teaching about right triangles and guiding students in learning to identify the hypotenuse, it may be necessary to provide additional support in addressing the differences among the terms *long, longer,* and *longest*, as comparative words like these (with the same root) may not yet be familiar to English language learners.

Classwork

Opening (5 minutes)

MP.3

Show students the three triangles below. Give students the direction to determine as much as they can about the triangles. If necessary, give the direction to apply the Pythagorean theorem, in particular. Then, have a discussion with students about their recollection of the theorem. Basic points should include the theorem, the converse of the theorem, and the fact that when the theorem leads them to an answer of the form $c^2 = x^2$, then $c = x$ (perfect squares).



In the first triangle, students are required to use the Pythagorean theorem to determine the unknown side length. Let us call the unknown side length x cm. Then, x is 8 cm because, by the Pythagorean theorem, $17^2 - 15^2 = x^2$, and $64 = x^2$. Since 64 is a perfect square, then students should identify the length of x as 8 cm.

In the second triangle, students are required to use the converse of the Pythagorean theorem to determine that it is a right triangle.

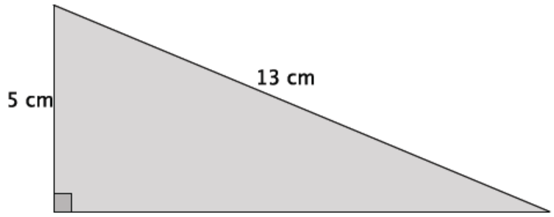
In the third triangle, students are required to use the converse of the Pythagorean theorem to determine that it is not a right triangle.

Example 1 (3 minutes)

- Recall the Pythagorean theorem and its converse for right triangles.
  - *The Pythagorean theorem states that a right triangle with leg lengths a and b and hypotenuse c will satisfy $a^2 + b^2 = c^2$. The converse of the theorem states that if a triangle with side lengths a, b, and c satisfies the equation $a^2 + b^2 = c^2$, then the triangle is a right triangle.*

Example 1

Write an equation that will allow you to determine the length of the unknown side of the right triangle.



Scaffolding:

Consider using cutouts of the triangles in this lesson to further illustrate the difference between triangles with whole number hypotenuses and those without. Then, call on students to measure the lengths directly for Examples 1–3. Cutouts drawn to scale are provided at the end of the lesson.

- Write an equation that will allow you to determine the length of the unknown side of the right triangle.

Note: Students may use a different symbol to represent the unknown side length.

  - *Let b represent the unknown side length*. Then, $5^2 + b^2 = 13^2$.

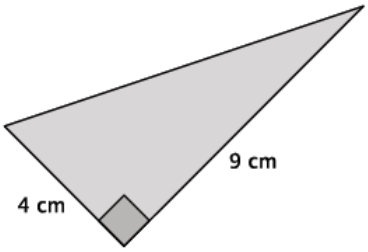
Verify that students wrote the correct equation; then, allow them to solve it. Ask them how they knew the correct answer was 12. They should respond that $13^2 - 5^2 = 144$, and since 144 is a perfect square, they knew that the unknown side length must be 12 cm.

Example 2 (5 minutes)

MP.1

Example 2

Write an equation that will allow you to determine the length of the unknown side of the right triangle.



- Write an equation that will allow you to determine the length of the unknown side of the right triangle.
  - *Let c represent the length of the hypotenuse. Then,* $4^2 + 9^2 = c^2$.

MP.1

- There is something different about this triangle. What is the length of the missing side? If you cannot find the length of the missing side exactly, then find a good approximation.

Provide students time to find an approximation for the length of the unknown side. Select students to share their answers and explain their reasoning. Use the points below to guide their thinking as needed.

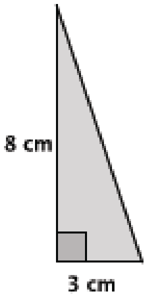
- How is this problem different from the last one?
  - *The answer is* $c^2 = 97$. *Since* 97 *is not a perfect square, the exact length cannot be represented as an integer.*
- Since 97 is not a perfect square, we cannot determine the exact length of the hypotenuse as an integer; however, we can make an estimate. Think about all of the perfect squares we have seen and calculated in past discussions. The number 97 is between which two perfect squares?
  - *The number* 97 *is between* 81 *and* 100*.*
- If the length of the hypotenuse were $c^2 = 81$, what would be the length of the hypotenuse?
  - *The length would be* 9 cm.
- If the length of the hypotenuse were $c^2 = 100$, what would be the length of the hypotenuse?
  - *The length would be* 10 cm.
- At this point, we know that the length of the hypotenuse is somewhere between 9 cm and 10 cm. Think about the length to which it is closest. The actual length of the hypotenuse is $c^2 = 97$. To which perfect square number, 100 or 81, is 97 closer?
  - *The number* 97 *is closer to the perfect square* 100 *than to the perfect square* 81*.*
- Now that we know that the length of the hypotenuse of this right triangle is between 9 cm and 10 cm, but closer to 10 cm, let's try to get an even better estimate of the length. Choose a number between 9 and 10 but closer to 10. Square that number. Do this a few times to see how close you can get to the number 97.

Provide students time to check a few numbers between 9 and 10. Students should see that the length is somewhere between 9.8 and 9.9 because $9.8^2 = 96.04$ and $9.9^2 = 98.01$. Some students may even check 9.85; $9.85^2 = 97.0225$. This activity will show students that an estimation of the length being between 9 cm and 10 cm is indeed accurate, and it will help students develop an intuitive sense of how to estimate square roots.

Example 3 (4 minutes)

Example 3

Write an equation to determine the length of the unknown side of the right triangle.



- Write an equation to determine the length of the unknown side of the right triangle.
  - *Let c represent the length of the hypotenuse. Then,* $3^2 + 8^2 = c^2$.

Verify that students wrote the correct equation, and then allow them to solve it. Instruct them to estimate the length, if necessary. Then, let them continue to work. When most students have finished, ask the questions below.

- Could you determine an answer for the length of the hypotenuse as an integer?
  - *No. The square of the length of the hypotenuse,* $c^2 = 73$, *is not a perfect square.*

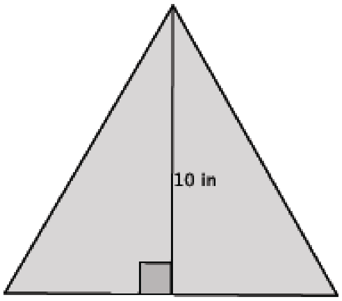
Optionally, you can ask, "Can anyone find the exact length of side c, as a rational number?" It is important that students recognize that no one can determine the exact length of the hypotenuse as a rational number at this point.

- Since 73 is not a perfect square, we cannot determine the exact length of the hypotenuse as a whole number. Let's estimate the length. Between which two whole numbers is the length of the hypotenuse? Explain.
  - *Since* 73 *is between the two perfect squares* 64 *and* 81, *we know the length of the hypotenuse must be between* 8 cm *and* 9 cm.
- Is the length closer to 8 cm or 9 cm? Explain.
  - *The length is closer to* 9 cm *because* 73 *is closer to* 81 *than it is to* 64.
- The length of the hypotenuse is between 8 cm and 9 cm but closer to 9 cm.

Example 4 (8 minutes)

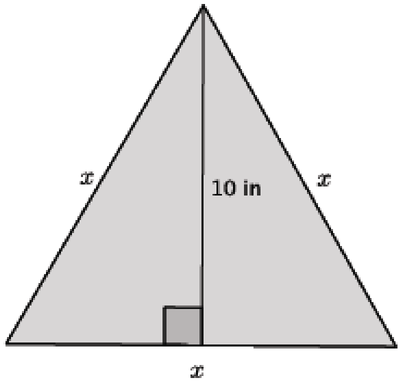
Example 4

In the figure below, we have an equilateral triangle with a height of 10 inches. What do we know about an equilateral triangle?



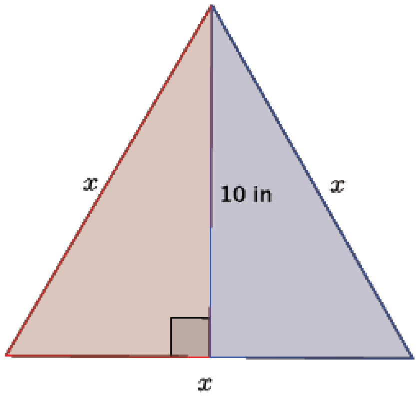
- In the figure below, we have an equilateral triangle with a height of 10 inches. What do we know about an equilateral triangle?
  - *Equilateral triangles have sides that are all of the same length and angles that are all of the same degree, namely* $60°$.

- Let’s say the length of the sides is x inches. Determine the approximate length of the sides of the triangle.



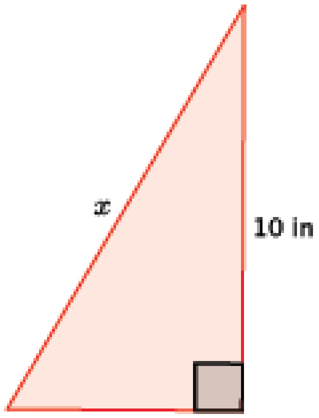
- What we actually have here are two congruent right triangles.

Trace one of the right triangles on a transparency, and reflect across the line representing the height of the triangle to convince students that an equilateral triangle is composed of two congruent right triangles.



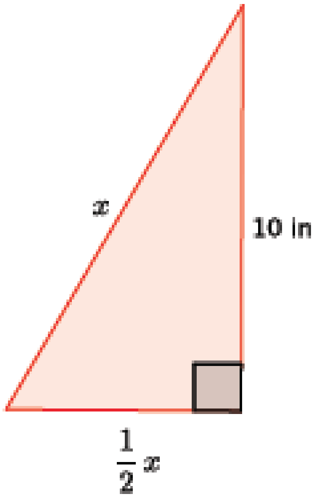
MP.1

- With this knowledge, we need to determine the length of the base of one of the right triangles. If we know that the length of the base of the equilateral triangle is x, then what is the length of the base of one of the right triangles? Explain.



  - *The length of the base of one of the right triangles must be $\frac{1}{2}x$ because the equilateral triangle has a base of length x. Since the equilateral triangle is composed of two congruent right triangles, we know that the base of each of the right triangles is of the same length (reflections preserve lengths of segments). Therefore, each right triangle has a base length of $\frac{1}{2}x$.*

- Now that we know the length of the base of the right triangle, write an equation for this triangle using the Pythagorean theorem.



  - $\left(\frac{1}{2}x\right)^2 + 10^2 = x^2$

MP.1

Verify that students wrote the correct equation, and then ask students to explain the meaning of each term of the equation. Allow students time to solve the equation in pairs or small groups. Instruct them to make an estimate of the length, if necessary. Then, let them continue to work. When most students have finished, continue with the discussion below.

- Explain your thinking about this problem. What did you do with the equation $\left(\frac{1}{2}x\right)^2 + 10^2 = x^2$?

If students are stuck, ask them questions that help them work through the computations below. For example, you can ask them what they recall about the laws of exponents to simplify the term $\left(\frac{1}{2}x\right)^2$ or how to use the properties of equality to get the answer in the form of x^2 equal to a constant.

  - *We had to solve for x:*

$$\left(\frac{1}{2}x\right)^2 + 10^2 = x^2$$
$$\frac{1}{4}x^2 + 100 = x^2$$
$$\frac{1}{4}x^2 - \frac{1}{4}x^2 + 100 = x^2 - \frac{1}{4}x^2$$
$$100 = \frac{3}{4}x^2$$
$$\frac{400}{3} = x^2$$
$$133.3 \approx x^2$$

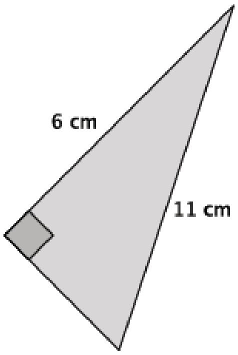
- Now that we know that $x^2 \approx 133.3$, find a whole number estimate for the length of x. Explain your thinking.
    - *The length of x is approximately* 12 in. *The number* 133.3 *is between the perfect squares* 121 *and* 144. *Since* 133.3 *is closer to* 144 *than* 121, *we know that the value of x is between* 11 *and* 12 *but closer to* 12.

Exercises 1–3 (7 minutes)

Students complete Exercises 1–3 independently.

Exercises

1. Use the Pythagorean theorem to find a whole number estimate of the length of the unknown side of the right triangle. Explain why your estimate makes sense.

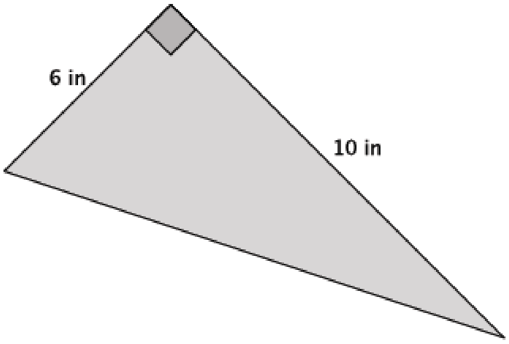


Let x be the length of the unknown side.

$$6^2 + x^2 = 11^2$$
$$36 + x^2 = 121$$
$$x^2 = 85$$

The length of the unknown side of the triangle is approximately 9 cm. The number 85 is between the perfect squares 81 and 100. Since 85 is closer to 81 than 100, then the length of the unknown side of the triangle is closer to 9 than it is to 10.

2. Use the Pythagorean theorem to find a whole number estimate of the length of the unknown side of the right triangle. Explain why your estimate makes sense.

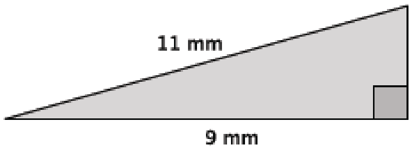


Let c be the length of the hypotenuse.

$$6^2 + 10^2 = c^2$$
$$36 + 100 = c^2$$
$$136 = c^2$$

The length of the hypotenuse is approximately 12 in. The number 136 is between the perfect squares 121 and 144. Since 136 is closer to 144 than 121, then the length of the unknown side of the triangle is closer to 12 than it is to 11.

3. Use the Pythagorean theorem to find a whole number estimate of the length of the unknown side of the right triangle. Explain why your estimate makes sense.



Let x be the length of the unknown side.

$$9^2 + x^2 = 11^2$$
$$81 + x^2 = 121$$
$$x^2 = 40$$

The length of the hypotenuse is approximately 6 mm. The number 40 is between the perfect squares 36 and 49. Since 40 is closer to 36 than 49, then the length of the unknown side of the triangle is closer to 6 than it is to 7.

Discussion (3 minutes)

- Our estimates for the lengths in the problems in this lesson are acceptable, but we can do better. Instead of saying that a length is between two particular whole numbers and closer to one compared to the other, we will soon learn how to make more precise estimates.
- Obviously, since the lengths have been between two integers (e.g., between 8 and 9), we will need to look at the numbers between the integers: the rational numbers (fractions). That means we will need to learn more about rational numbers and all numbers between the integers on the number line, in general.
- The examination of those numbers will be the focus of the next several lessons.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know what a perfect square is.
- We know that when the square of the length of an unknown side of a triangle is not equal to a perfect square, we can estimate the side length by determining which two perfect squares the square of the length is between.
- We know that we will need to look more closely at the rational numbers in order to make better estimates of the lengths of unknown sides of a right triangle.

Lesson Summary

Perfect square numbers are those that are a product of an integer factor multiplied by itself. For example, the number 25 is a perfect square number because it is the product of 5 multiplied by 5.

When the square of the length of an unknown side of a right triangle is not equal to a perfect square, you can estimate the length as a whole number by determining which two perfect squares the square of the length is between.

Example:

Let c represent the length of the hypotenuse. Then,

$$\begin{aligned} 3^2 + 7^2 &= c^2 \\ 9 + 49 &= c^2 \\ 58 &= c^2. \end{aligned}$$

The number 58 is not a perfect square, but it is between the perfect squares 49 and 64. Therefore, the length of the hypotenuse is between 7 and 8 but closer to 8 because 58 is closer to the perfect square 64 than it is to the perfect square 49.

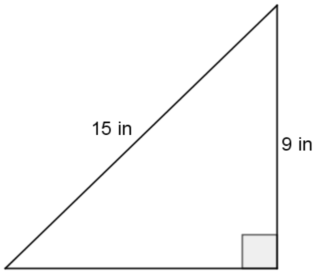
Exit Ticket (5 minutes)

Name ____ Date ____________________

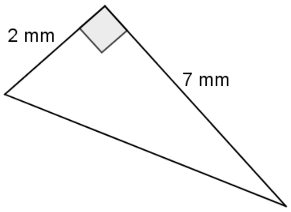
Lesson 1: The Pythagorean Theorem

Exit Ticket

1. Determine the length of the unknown side of the right triangle. If you cannot determine the length exactly, then determine which two integers the length is between and the integer to which it is closest.

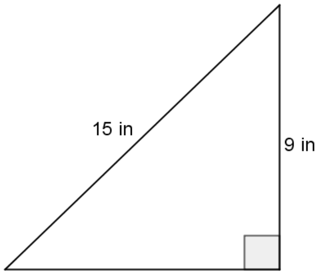


2. Determine the length of the unknown side of the right triangle. If you cannot determine the length exactly, then determine which two integers the length is between and the integer to which it is closest.



Exit Ticket Sample Solutions

1. **Determine the length of the unknown side of the right triangle. If you cannot determine the length exactly, then determine which two integers the length is between and the integer to which it is closest.**

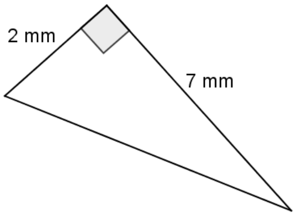


Let x be the length of the unknown side.

$$9^2 + x^2 = 15^2$$
$$81 + x^2 = 225$$
$$x^2 = 144$$
$$x = 12$$

The length of the unknown side is 12 in. The Pythagorean theorem led me to the fact that the square of the unknown side is 144. We know 144 is a perfect square, and 144 is equal to 12^2; therefore, $x = 12$, and the unknown length of the triangle is 12 in.

2. **Determine the length of the unknown side of the right triangle. If you cannot determine the length exactly, then determine which two integers the length is between and the integer to which it is closest.**



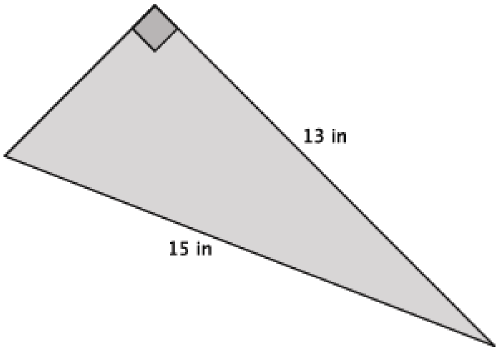
Let x be the length of the unknown side.

$$2^2 + 7^2 = x^2$$
$$4 + 49 = x^2$$
$$53 = x^2$$

The number 53 is between the perfect squares 49 and 64. Since 53 is closer to 49 than 64, then the length of the unknown side of the triangle is closer to 7 than 8.

Problem Set Sample Solutions

1. **Use the Pythagorean theorem to estimate the length of the unknown side of the right triangle. Explain why your estimate makes sense.**

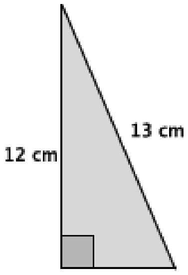


Let x be the length of the unknown side.

$$13^2 + x^2 = 15^2$$
$$169 + x^2 = 225$$
$$x^2 = 56$$

The number 56 is between the perfect squares 49 and 64. Since 56 is closer to 49 than it is to 64, then the length of the unknown side of the triangle is closer to 7 than it is to 8.

2. **Use the Pythagorean theorem to estimate the length of the unknown side of the right triangle. Explain why your estimate makes sense.**

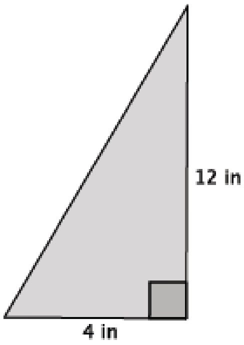


Let x be the length of the unknown side.

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 = 25$$
$$x = 5$$

The length of the unknown side is 5 cm. The Pythagorean theorem led me to the fact that the square of the unknown side is 25. Since 25 is a perfect square, 25 is equal to 5^2; therefore, $x = 5$, and the unknown length of the triangle is 5 cm.

3. **Use the Pythagorean theorem to estimate the length of the unknown side of the right triangle. Explain why your estimate makes sense.**

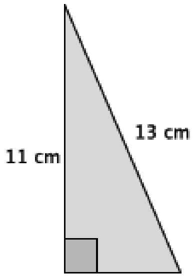


Let c be the length of the hypotenuse.

$$4^2 + 12^2 = c^2$$
$$16 + 144 = c^2$$
$$160 = c^2$$

The number 160 is between the perfect squares 144 and 169. Since 160 is closer to 169 than it is to 144, then the length of the hypotenuse of the triangle is closer to 13 than it is to 12.

4. **Use the Pythagorean theorem to estimate the length of the unknown side of the right triangle. Explain why your estimate makes sense.**

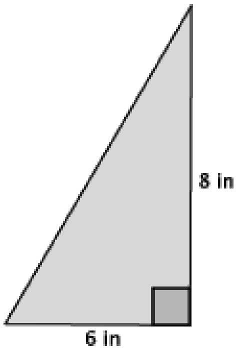


Let x be the length of the unknown side.

$$x^2 + 11^2 = 13^2$$
$$x^2 + 121 = 169$$
$$x^2 = 48$$

The number 48 is between the perfect squares 36 and 49. Since 48 is closer to 49 than it is to 36, then the length of the unknown side of the triangle is closer to 7 than it is to 6.

5. **Use the Pythagorean theorem to estimate the length of the unknown side of the right triangle. Explain why your estimate makes sense.**

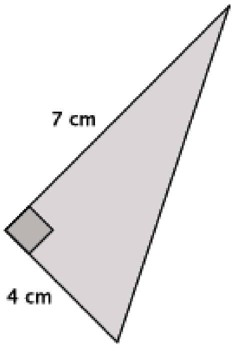


Let c be the length of the hypotenuse.

$$6^2 + 8^2 = c^2$$
$$36 + 64 = c^2$$
$$100 = c^2$$
$$10 = c$$

The length of the hypotenuse is 10 in. The Pythagorean theorem led me to the fact that the square of the unknown side is 100. We know 100 is a perfect square, and 100 is equal to 10^2; therefore, $c = 10$, and the length of the hypotenuse of the triangle is 10 in.

6. **Determine the length of the unknown side of the right triangle. Explain how you know your answer is correct.**



Let c be the length of the hypotenuse.

$$7^2 + 4^2 = c^2$$
$$49 + 16 = c^2$$
$$65 = c^2$$

The number 65 is between the perfect squares 64 and 81. Since 65 is closer to 64 than it is to 81, then the length of the hypotenuse of the triangle is closer to 8 than it is to 9.

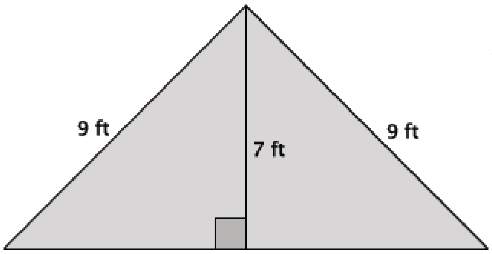
7. **Use the Pythagorean theorem to estimate the length of the unknown side of the right triangle. Explain why your estimate makes sense.**

Let x be the length of the unknown side.

$$3^2 + x^2 = 12^2$$
$$9 + x^2 = 144$$
$$x^2 = 135$$

The number 135 is between the perfect squares 121 and 144. Since 135 is closer to 144 than it is to 121, then the length of the unknown side of the triangle is closer to 12 than it is to 11.

8. **The triangle below is an isosceles triangle. Use what you know about the Pythagorean theorem to determine the approximate length of the base of the isosceles triangle.**

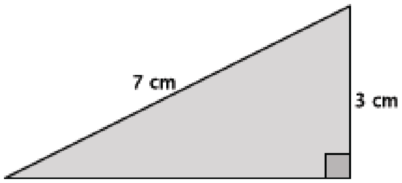


Let x represent the length of the base of one of the right triangles of the isosceles triangle.

$$x^2 + 7^2 = 9^2$$
$$x^2 + 49 = 81$$
$$x^2 = 32$$

Since 32 is between the perfect squares 25 and 36 but closer to 36, the approximate length of the base of the right triangle is 6 ft. Since there are two right triangles, then the length of the base of the isosceles triangle is approximately 12 ft.

9. **Give an estimate for the area of the triangle shown below. Explain why it is a good estimate.**



Let x represent the length of the base of the right triangle.

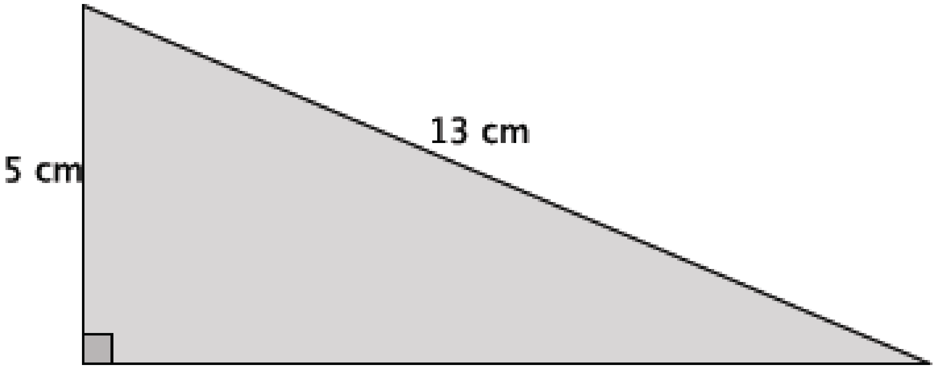
$$x^2 + 3^2 = 7^2$$
$$x^2 + 9 = 49$$
$$x^2 = 40$$

Since 40 is between the perfect squares 36 and 49 but closer to 36, the approximate length of the base is 6 cm. So, the approximate area of the triangle is

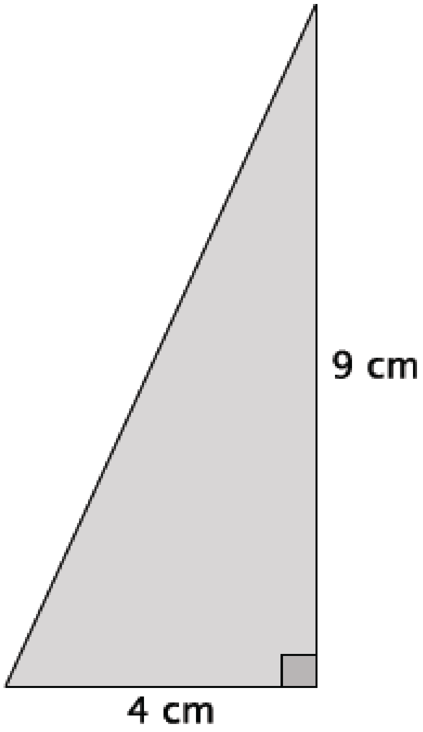
$$\frac{6(3)}{2} = 9.$$

9 cm^2 is a good estimate because of the approximation of the length of the base. Further, since the hypotenuse is the longest side of the right triangle, approximating the length of the base as 6 cm makes mathematical sense because it has to be shorter than the hypotenuse.

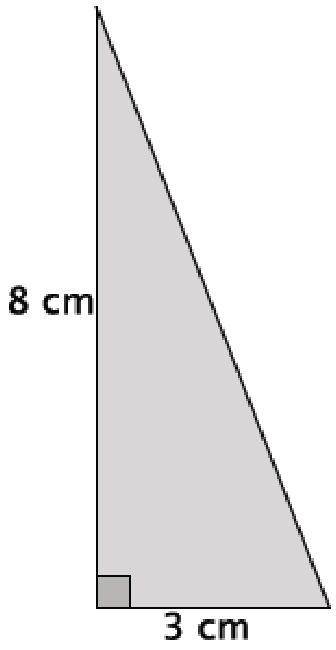
Example 1



Example 2



Example 3



Lesson 2: Square Roots

Student Outcomes

- Students know that for most whole numbers n, n is not a perfect square, and they understand the square root symbol, $\sqrt{n}$. Students find the square root of small perfect squares.
- Students approximate the location of square roots of whole numbers on the number line.

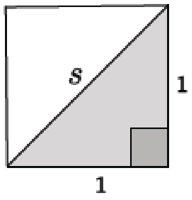
Classwork

Discussion (10 minutes)

MP.1

As an option, the discussion can be framed as a challenge. Distribute compasses, and ask students, "How can we determine an estimate for the length of the diagonal of the unit square?"

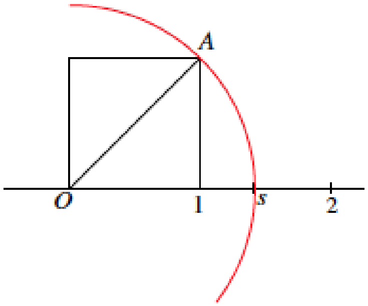
- Consider a unit square, a square with side lengths equal to 1. How can we determine the length of the diagonal, s, of the unit square?



  - *We can use the Pythagorean theorem to determine the length of the diagonal.*

$$1^2 + 1^2 = s^2$$
$$2 = s^2$$

- What number, s, times itself is equal to 2?
  - *We don't know exactly, but we know the number has to be between* 1 *and* 2.
- We can show that the number must be between 1 and 2 if we place the unit square on a number line as shown. Then, consider a circle with center O and radius equal to the length of the hypotenuse of the triangle OA.



Scaffolding:

Depending on students' experience, it may be useful to review or teach the concept of square numbers and perfect squares.

- We can see that the length OA is somewhere between 1 and 2 but precisely at point s. But what is that number s?

- From our work with the Pythagorean theorem, we know that 2 is not a perfect square. Thus, the length of the diagonal must be between the two integers 1 and 2, and that is confirmed on the number line. To determine the number s, we should look at that part of the number line more closely. To do so, we need to discuss what kinds of numbers lie between the integers on a number line. What do we already know about those numbers?

Lead a discussion about the types of numbers found between the integers on a number line. Students should identify that rational numbers, such as fractions and decimals, lie between the integers. Have students give concrete examples of numbers found between the integers 1 and 2. Consider asking students to write a rational number, x, so that $1 < x < 2$, on a sticky note and then to place it on a number line drawn on a poster or white board. At the end of this part of the discussion, make clear that all of the numbers students identified are rational and in the familiar forms of fractions, mixed numbers, and decimals. Then, continue with the discussion below about square roots.

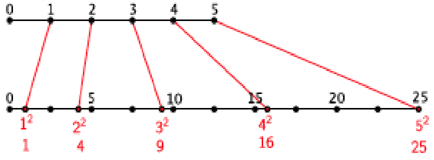
Scaffolding:

Students may benefit from an oral recitation of square roots of perfect squares here and throughout the module. Consider some repeated "quick practice," calling out, for example: "What is the square root of 81?" and "What is the square root of 100?" Ask for choral or individual responses.

- There are other numbers on the number line between the integers. Some of the square roots of whole numbers are equal to whole numbers, but most lie between the integers on the number line. A positive number whose square is equal to a positive number b is denoted by the symbol $\sqrt{b}$. The symbol $\sqrt{b}$ automatically denotes a positive number (e.g., $\sqrt{4}$ is always 2, not -2). The number $\sqrt{b}$ is called *a positive square root of* b. We will soon learn that it is *the* positive square root, i.e., there is only one.
- What is $\sqrt{25}$, i.e., the positive square root of 25? Explain.
  - *The positive square root of* 25 *is* 5 *because* $5^2 = 25$.
- What is $\sqrt{9}$, i.e., the positive square root of 9? Explain.
  - *The positive square root of* 9 *is* 3 *because* $3^2 = 9$.

Scaffolding:

If students are struggling with the concept of a square root, it may help to refer to visuals that relate numbers and their squares. Showing this visual:



and asking questions (e.g., "What is the square root of 9?") will build students' understanding of square roots through their understanding of squares.

Exercises 1–4 (5 minutes)

Students complete Exercises 1–4 independently.

Exercises 1–4

1. **Determine the positive square root of 81, if it exists. Explain.**

   ***The square root of* 81 *is* 9 *because* $9^2 = 81$.**

2. **Determine the positive square root of 225, if it exists. Explain.**

   ***The square root of* 225 *is* 15 *because* $15^2 = 225$.**

3. **Determine the positive square root of -36, if it exists. Explain.**

   The number* -36 *does not have a square root because there is no number squared that can produce a negative number.

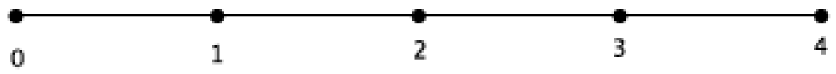
4. **Determine the positive square root of 49, if it exists. Explain.**

   ***The square root of* 49 *is* 7 *because* $7^2 = 49$.**

Discussion (15 minutes)

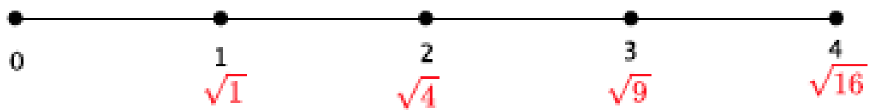
- Now, back to our unit square. We said that the length of the diagonal is s, and $s^2 = 2$. Now that we know about square roots, we can say that the length of $s = \sqrt{2}$ and that the number $\sqrt{2}$ is between integers 1 and 2. Let's look at the number line more generally to see if we can estimate the value of $\sqrt{2}$.
- Take a number line from 0 to 4:

Discussion



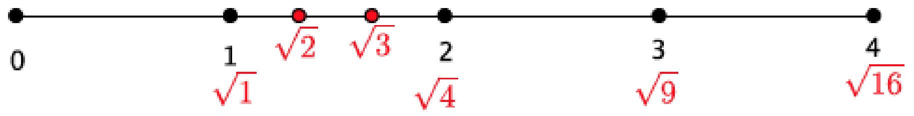
- Place the numbers $\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, and $\sqrt{16}$ on the number line, and explain how you knew where to place them.

Solutions are shown below in red.



- Place the numbers $\sqrt{2}$ and $\sqrt{3}$ on the number line. Be prepared to explain your reasoning.

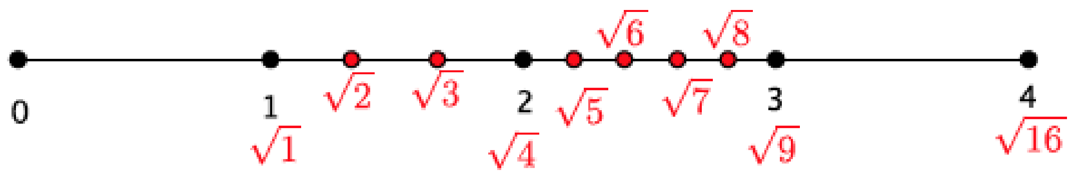
Solutions are shown below in red. Students should reason that the numbers $\sqrt{2}$ and $\sqrt{3}$ belong on the number line between $\sqrt{1}$ and $\sqrt{4}$. They could be more specific by saying that if you divide the segment between integers 1 and 2 into three equal parts, then $\sqrt{2}$ would be at the first division, $\sqrt{3}$ would be at the second division, and $\sqrt{4}$ is already at the third division, 2, on the number line. Given that reasoning, students should be able to estimate the value of $\sqrt{2} \approx 1\frac{1}{3}$.



MP.3

- Place the numbers $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, and $\sqrt{8}$ on the number line. Be prepared to explain your reasoning.

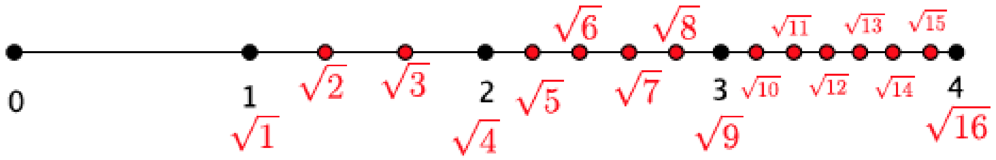
Solutions are shown below in red. The discussion about placement should be similar to the previous one.



MP.3

- Place the numbers $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$, $\sqrt{13}$, $\sqrt{14}$, and $\sqrt{15}$ on the number line. Be prepared to explain your reasoning.

Solutions are shown below in red. The discussion about placement should be similar to the previous one.



- Our work on the number line shows that there are many more square roots of whole numbers that are not perfect squares than those that are perfect squares. On the number line above, we have four perfect square numbers and twelve that are not! After we do some more work with roots, in general, we will cover exactly how to describe these numbers and how to approximate their values with greater precision. For now, we will estimate their locations on the number line using what we know about perfect squares.

Exercises 5–9 (5 minutes)

Students complete Exercises 5–9 independently. Calculators may be used for approximations.

Exercises 5–9

Determine the positive square root of the number given. If the number is not a perfect square, determine which whole number the square root would be closest to, and then use *guess and check* to give an approximate answer to one or two decimal places.

5. $\sqrt{49}$

7

6. $\sqrt{62}$

The square root of 62 *is close to* 8*. The square root of* 62 *is approximately* 7.9 *because* $7.9^2 = 62.41$.

7. $\sqrt{122}$

The square root of 122 *is close to* 11*. Students may guess a number between* 122 *and* 122.1 *because* $11.05^2 = 122.1025$.

8. $\sqrt{400}$

20

9. **Which of the numbers in Exercises 5–8 are not perfect squares? Explain.**

The numbers 62 *and* 122 *are not perfect squares because there is no integer* x *to satisfy* $x^2 = 62$ *or* $x^2 = 122$.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that there are numbers on the number line between the integers. The ones we looked at in this lesson are square roots of whole numbers that are not perfect squares.
- We know that when a positive number x is squared and the result is b, then $\sqrt{b}$ is equal to x.
- We know how to approximate the square root of a whole number and its location on a number line by figuring out which two perfect squares it is between.

Lesson Summary

A positive number whose square is equal to a positive number b is denoted by the symbol $\sqrt{b}$. The symbol $\sqrt{b}$ automatically denotes a positive number. For example, $\sqrt{4}$ is always 2, not -2. The number $\sqrt{b}$ is called *a positive square root of b*.

The square root of a perfect square of a whole number is that whole number. However, there are many whole numbers that are not perfect squares.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 2: Square Roots

Exit Ticket

1. Write the positive square root of a number x in symbolic notation.

2. Determine the positive square root of 196, if it exists. Explain.

3. Determine the positive square root of 50, if it exists. Explain.

4. Place the following numbers on the number line: $\sqrt{16}$, $\sqrt{9}$, $\sqrt{11}$, and 3.5.

Exit Ticket Sample Solutions

1. **Write the square root of a number x in symbolic notation.**

   $\sqrt{x}$

2. **Determine the positive square root of 196, if it exists. Explain.**

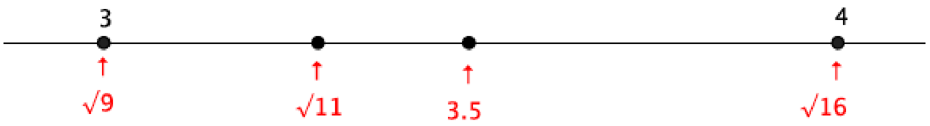
   $\sqrt{196} = 14$ *because* $14^2 = 196$.

3. **Determine the positive square root of 50, if it exists. Explain.**

   $\sqrt{50}$ *is between* 7 *and* 8 *but closer to* 7. *The reason is that* $7^2 = 49$, *and* $8^2 = 64$. *The number* 50 *is between* 49 *and* 64 *but closer to* 49. *Therefore, the square root of* 50 *is close to* 7.

4. **Place the following numbers on the number line: $\sqrt{16}$, $\sqrt{9}$, $\sqrt{11}$, and 3.5.**

   Solutions are shown in red below.



Problem Set Sample Solutions

Determine the positive square root of the number given. If the number is not a perfect square, determine the integer to which the square root would be closest.

1. $\sqrt{169}$

   13

2. $\sqrt{256}$

   16

3. $\sqrt{81}$

   9

4. $\sqrt{147}$

   The number 147 *is not a perfect square. It is between the perfect squares* 144 *and* 169 *but closer to* 144. *Therefore, the square root of* 147 *is close to* 12.

5. $\sqrt{8}$

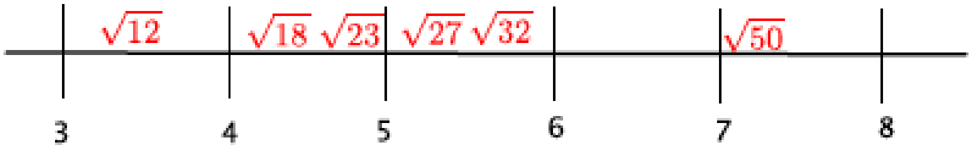
   The number 8 *is not a perfect square. It is between the perfect squares* 4 *and* 9 *but closer to* 9. *Therefore, the square root of* 8 *is close to* 3.

6. **Which of the numbers in Problems 1–5 are not perfect squares? Explain.**

   The numbers 147 *and* 8 *are not perfect squares because there is no integer* x *so that* $x^2 = 147$ *or* $x^2 = 8$.

7. **Place the following list of numbers in their approximate locations on a number line.**

   $\sqrt{32}$, $\sqrt{12}$, $\sqrt{27}$, $\sqrt{18}$, $\sqrt{23}$, and $\sqrt{50}$



   Answers are noted in red.

8. **Between which two integers will $\sqrt{45}$ be located? Explain how you know.**

   The number 45 *is not a perfect square. It is between the perfect squares* 36 *and* 49 *but closer to* 49. *Therefore, the square root of* 45 *is between the integers* 6 *and* 7 *because* $\sqrt{36} = 6$ *and* $\sqrt{49} = 7$ *and* $\sqrt{36} < \sqrt{45} < \sqrt{49}$.

Lesson 3: Existence and Uniqueness of Square Roots and Cube Roots

Student Outcomes

- Students know that the positive square root and the cube root exist for all positive numbers and both a square root of a number and a cube root of a number are unique.
- Students solve simple equations that require them to find the square root or cube root of a number.

Lesson Notes

This lesson has two options for showing the existence and uniqueness of positive square roots and cube roots. Each option has an Opening Exercise and a Discussion that follows. The first option has students explore facts about numbers on a number line, leading to an understanding of the trichotomy law, followed by a discussion of how the law applies to squares of numbers, which should give students a better understanding of what square and cube roots are and how they are unique. The second option explores numbers and their squares via a Find the Rule exercise, followed by a discussion that explores how square and cube roots are unique. The first option includes a discussion of the basic inequality property, a property referred to in subsequent lessons. The basic inequality property states that if x, y, w, and z are positive numbers so that $x < y$ and $w < z$, then $xw < yz$. Further, if $x = w$ and $y = z$, when $x < y$, then $x^2 < y^2$. Once the first or second option is completed, the lesson continues with a discussion of how to solve equations using square roots.

Throughout this and subsequent lessons, we ask students to find only the positive values of x that satisfy a radical equation. The reason is that in Algebra I students will solve radical equations by setting the equation equal to zero and then factoring the quadratic to find the solutions:

$$
\begin{aligned}
x^2 &= 25 \\
x^2 - 25 &= 0 \\
(x+5)(x-5) &= 0 \\
x &= \pm 5.
\end{aligned}
$$

At this point, students have not learned how to factor quadratics and will solve all equations using the square root symbol, which means students are only responsible for finding the positive solution(s) to an equation.

Classwork

Opening (5 minutes): Option 1

Ask students the following to prepare for the discussion that follows.

- Considering only the positive integers, if $x^2 = 4$, what must x equal? Could x be any other number?
  - *The number $x = 2$ and no other number.*

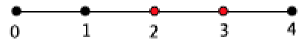
- If $c = 3$ and $d = 4$, compare the numbers c^2 and d^2 and the numbers c and d.
  - $c^2 < d^2$ and $c < d$.
- If $c < d$, could $c = d$? Explain.
  - *By definition if $c < d$, then $c \neq d$. Because $c < d$, c will be to the left of d on the number line, which means c and d are not at the same point on the number line. Therefore, $c \neq d$.*
- If $c < d$, could $c > d$?
  - *By definition if $c < d$, c will be to the left of d on the number line. The inequality $c > d$ means that c would be to the right of d on the number line. If $c < d$, then $c > d$ cannot also be true because c cannot simultaneously be to the right and to the left of d.*

Discussion (12 minutes): Option 1

(An alternative discussion is provided below.) Once this discussion is complete, continue with the discussion on page 39.

Scaffolding:

The number line can help students make sense of the trichotomy law. Give students two numbers, and ask them which of the three possibilities for those numbers is true. In each case, only one of three is true for any pair of numbers. For example, given the numbers $c = 2$ and $d = 3$, which of the following is true: $c = d$, $c < d$, or $c > d$?



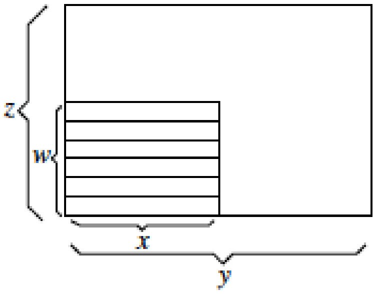
- We will soon be solving equations that include roots. For this reason, we want to be sure that the answer we get when we simplify is correct. Specifically, we want to be sure that we can get an answer, that it exists, that the answer we get is correct, and that it is unique to the given situation.
- To this end, existence requires us to show that given a positive number b and a positive integer n, there is one and only one positive number c, so that when $c^n = b$, we say "c is the positive n^{th} root of b." When $n = 2$, we say "c is the positive square root of b" and when $n = 3$, we say "c is the positive cube root of b." Uniqueness requires us to show that given two positive numbers c and d and $n = 2$: If $c^2 = d^2$, then $c = d$. This statement implies uniqueness because both c and d are the positive square root of b, i.e., $c^2 = b$ and $d^2 = b$; since $c^2 = d^2$, then $c = d$. Similarly, when $n = 3$, if $c^3 = d^3$, then $c = d$. The reasoning is the same since both c and d are the positive cube root of b, i.e., $c^3 = b$ and $d^3 = b$; since $c^3 = d^3$, then $c = d$. Showing uniqueness will also show existence, so we will focus on proving the uniqueness of square and cube roots. **MP.3**
- To show that $c = d$, we will use the trichotomy law. The trichotomy law states that given two numbers c and d, one and only one of the following three possibilities is true.

  (i) $c = d$

  (ii) $c < d$

  (iii) $c > d$

  We will show $c = d$ by showing that $c < d$ and $c > d$ cannot be true.
- If x, y, w, and z are positive numbers so that $x < y$ and $w < z$, is it true that $xw < yz$? Explain.
  - *Yes, it is true that $xw < yz$. Since all of the numbers are positive and both x and w are less than y and z, respectively, then their product must also be less. For example, since $3 < 4$, and $5 < 6$, then $3 \times 5 < 4 \times 6$.*



- This basic inequality property can also be explained in terms of areas of a rectangle. The picture below clearly shows that when $x < y$ and $w < z$, then $xw < yz$.
- We will use this fact to show that $c < d$ and $c > d$ cannot be true. We begin with $c^n < d^n$ when $n = 2$. By the basic inequality, $c^2 < d^2$. Now, we look at the case where $n = 3$. We will use the fact that $c^2 < d^2$ to show $c^3 < d^3$. What can we do to show $c^3 < d^3$?
    - *We can multiply c^2 by c and d^2 by d. The basic inequality guarantees that since $c < d$ and $c^2 < d^2$, that $c^2 \times c < d^2 \times d$, which is the same as $c^3 < d^3$.*

MP.3

- Using $c^3 < d^3$, how can we show $c^4 < d^4$?
    - *We can multiply c^3 by c and d^3 by d. The basic inequality guarantees that since $c < d$ and $c^3 < d^3$, $c^3 \times c < d^3 \times d$, which is the same as $c^4 < d^4$.*
- We can use the same reasoning for any positive integer n. We can use similar reasoning to show that if $c > d$, then $c^n > d^n$ for any positive integer n.
- Recall that we are trying to show that if $c^n = d^n$, then $c = d$ for $n = 2$ or $n = 3$. If we assume that $c < d$, then we know that $c^n < d^n$, which contradicts our hypothesis of $c^n = d^n$. By the same reasoning, if $c > d$, then $c^n > d^n$, which is also a contradiction of the hypothesis. By the trichotomy law, the only possibility left is that $c = d$. Therefore, we have shown that the square root or cube root of a number is unique and also exists.

Opening (8 minutes): Option 2

Begin by having students *find the rule* given numbers in two columns. The goal is for students to see the relationship between the square of a number and its square root and the cube of a number and its cube root. Students have to figure out the rule, then find missing values in the columns, and explain their reasoning. Provide time for students to do this independently. If necessary, allow students to work in pairs.

MP.8

- The numbers in each column are related. Your goal is to determine how they are related, determine which numbers belong in the blank parts of the columns, and write an explanation for how you know the numbers belong there.

Opening

Find the Rule Part 1

1	1
2	4
3	9
9	81
11	121
15	225
7	49
10	100
12	144
13	169
m	m^2
$\sqrt{n}$	n

Note: Students will not know how to write the cube root of a number using the proper notation, but it will be a good way to launch into the discussion below.

MP.8

Find the Rule Part 2

1	1
2	8
3	27
5	125
6	216
11	$1,331$
4	64
10	$1,000$
7	343
14	$2,744$
p	p^3
$\sqrt[3]{q}$	q

Discussion (9 minutes): Option 2

Once the Find the Rule exercise is finished, use the discussion points below; then, continue with the Discussion that follows on page 39.

- For Find the Rule Part 1, how were you able to determine which number belonged in the blank?
  - *To find the numbers that belonged in the blanks in the right column, I had to square the number in the left column. To find the numbers that belonged in the left column, I had to take the square root of the number in the right column.*
- When given the number m in the left column, how did we know the number that belonged to the right?
  - *Given m on the left, the number that belonged on the right was m^2.*
- When given the number n in the right column, how did we know the number that belonged to the left?
  - *Given n on the right, the number that belonged on the left was $\sqrt{n}$.*
- For Find the Rule Part 2, how were you able to determine which number belonged in the blank?
  - *To find the number that belonged in the blank in the right column, I had to multiply the number in the left column by itself three times. To find the number that belonged in the left column, I had to figure out which number multiplied by itself three times equaled the number that was in the right column.*
- When given the number p in the left column, how did we note the number that belonged to the right?
  - *Given p on the left, the number that belonged on the right was p^3.*
- When given the number q in the right column, the notation we use to denote the number that belongs to the left is similar to the notation we used to denote the square root. Given the number q in the right column, we write $\sqrt[3]{q}$ on the left. The 3 in the notation shows that we must find the number that multiplied by itself 3 times is equal to q.

- Were you able to write more than one number in any of the blanks?
    - *No, there was only one number that worked.*
- Were there any blanks that could not be filled?
    - *No, in each case there was a number that worked.*
- For Find the Rule Part 1, you were working with squared numbers and square roots. For Find the Rule Part 2, you were working with cubed numbers and cube roots. Just like we have perfect squares, there are also perfect cubes. For example, 27 is a perfect cube because it is the product of 3^3. For Find the Rule Part 2, you cubed the number on the left to fill the blank on the right and took the cube root of the number on the right to fill the blank on the left.
- We could extend the Find the Rule exercise to include an infinite number of rows, and in each case, we would be able to fill the blanks. Therefore, we can say that positive square roots and cube roots exist. Because only one number worked in each of the blanks, we can say that the positive roots are unique.
- We must learn about square roots and cube roots to solve equations. The properties of equality allow us to add, subtract, multiply, and divide the same number to both sides of an equal sign. We want to extend the properties of equality to include taking the square root and taking the cube root of both sides of an equation.
- Consider the equality $25 = 25$. What happens when we take the square root of both sides of the equal sign? Do we get a true number sentence?
    - *When we take the square root of both sides of the equal sign, we get* $5 = 5$. *Yes, we get a true number sentence.*
- Consider the equality $27 = 27$. What happens when we take the cube root of both sides of the equal sign? Do we get a true number sentence?
    - *When we take the cube root of both sides of the equal sign, we get* $3 = 3$. *Yes, we get a true number sentence.*
- At this point, we only know that the properties of equality can extend to those numbers that are perfect squares and perfect cubes, but it is enough to allow us to begin solving equations using square and cube roots.

Discussion (8 minutes)

The properties of equality have been proven for rational numbers, which are central in school mathematics. As we begin to solve equations that require roots, we are confronted with the fact that we may be working with irrational numbers (which have not yet been defined for students). Therefore, we make the assumption that all of the properties of equality for rational numbers are also true for irrational numbers, i.e., the real numbers, as far as computations are concerned. This is sometimes called the fundamental assumption of school mathematics (FASM). In the discussion below, we reference n^{th} roots. You may choose to discuss square and cube roots only.

- In the past, we have determined the length of the missing side of a right triangle, x, when $x^2 = 25$. What is that value, and how did you get the answer?
    - *The value of* x *is* 5 *because* x^2 *means* $x \cdot x$. *Since* $5 \times 5 = 25$, x *must be* 5.
- If we did not know that we were trying to find the length of the side of a triangle, then the answer could also be -5 because $-5 \times -5 = 25$. However, because we were trying to determine the length of the side of a triangle, the answer must be positive because a length of -5 does not make sense.
- Now that we know that positive square roots exist and are unique, we can begin solving equations that require roots.

- When we solve equations that contain roots, we do what we do for all properties of equality, i.e., we apply the operation to both sides of the equal sign. In terms of solving a radical equation, if we assume x is positive, then

$$\begin{aligned} x^2 &= 25 \\ \sqrt{x^2} &= \sqrt{25} \\ x &= \sqrt{25} \\ x &= 5. \end{aligned}$$

- Explain the first step in solving this equation.
    - ▫ *The first step is to take the square root of both sides of the equation.*
- It is by definition that when we use the symbol $\sqrt{}$, it automatically denotes a positive number; therefore, the solution to this equation is 5. In Algebra I, you will learn how to solve equations of this form without using the square root symbol, which means the possible values for x can be both 5 and -5 because $5^2 = 25$ and $(-5)^2 = 25$, but for now, we will only look for the positive solution(s) to our equations.

Note: In Algebra I, students will solve equations of this form by setting the equation equal to zero, then factoring the quadratic to find the solutions:

$$\begin{aligned} x^2 &= 25 \\ x^2 - 25 &= 0 \\ (x+5)(x-5) &= 0 \\ x &= \pm 5. \end{aligned}$$

At this point, students have not learned how to factor quadratics and will solve all equations using the square root symbol, which means students are only responsible for finding the positive solution(s) to an equation. Make it clear to students that *for now,* we need only find the positive solutions; as they continue to learn more about nonlinear equations, they will need to find all of the possible solutions.

- Consider the equation $x^2 = 25^{-1}$. What is another way to write 25^{-1}?
    - ▫ *The number 25^{-1} is the same as $\frac{1}{25}$.*
- Again, assuming that x is positive, we can solve the equation as before.

$$\begin{aligned} x^2 &= 25^{-1} \\ x^2 &= \frac{1}{25} \\ \sqrt{x^2} &= \sqrt{\frac{1}{25}} \\ x &= \sqrt{\frac{1}{25}} \\ x &= \frac{1}{5} \end{aligned}$$

We know we are correct because $\left(\frac{1}{5}\right)^2 = \frac{1}{25} = 25^{-1}$.

- The symbol $\sqrt[n]{}$ is called a *radical*. An equation that contains that symbol is referred to as a *radical equation*. So far, we have only worked with square roots ($n = 2$). Technically, we would denote a square root as $\sqrt[2]{}$, but it is understood that the symbol $\sqrt{}$ alone represents a square root.
- When $n = 3$, then the symbol $\sqrt[3]{}$ is used to denote the cube root of a number. Since $x^3 = x \cdot x \cdot x$, then the cube root of x^3 is x, i.e., $\sqrt[3]{x^3} = x$.
- For what value of x is the equation $x^3 = 8$ true?
  - $$\begin{aligned} x^3 &= 8 \\ \sqrt[3]{x^3} &= \sqrt[3]{8} \\ x &= \sqrt[3]{8} \\ x &= 2 \end{aligned}$$
- The n^{th} root of a number is denoted by $\sqrt[n]{}$. In the context of our learning, we will limit our work with radicals to square and cube roots.

Exercises 1–9 (10 minutes)

Students complete Exercises 1–9 independently. Allow them to use a calculator to check their answers. Also consider showing students how to use the calculator to find the square root of a number.

Exercises

Find the positive value of x that makes each equation true. Check your solution.

1. $x^2 = 169$

a. **Explain the first step in solving this equation.**

The first step is to take the square root of both sides of the equation.

b. **Solve the equation, and check your answer.**

$$\begin{aligned} x^2 &= 169 \\ \sqrt{x^2} &= \sqrt{169} \\ x &= \sqrt{169} \\ x &= 13 \end{aligned}$$

Check:

$$\begin{aligned} 13^2 &= 169 \\ 169 &= 169 \end{aligned}$$

2. **A square-shaped park has an area of 324 ft^2. What are the dimensions of the park? Write and solve an equation.**

$$\begin{aligned} x^2 &= 324 \\ \sqrt{x^2} &= \sqrt{324} \\ x &= \sqrt{324} \\ x &= 18 \end{aligned}$$

Check:

$$\begin{aligned} 18^2 &= 324 \\ 324 &= 324 \end{aligned}$$

The square park is 18 ft. *in length and* 18 ft. *in width.*

3. $625 = x^2$

$$\begin{aligned} 625 &= x^2 \\ \sqrt{625} &= \sqrt{x^2} \\ \sqrt{625} &= x \\ 25 &= x \end{aligned}$$

Check:

$$\begin{aligned} 625 &= 25^2 \\ 625 &= 625 \end{aligned}$$

4. A cube has a volume of 27 in^3. What is the measure of one of its sides? Write and solve an equation.

$$\begin{aligned} 27 &= x^3 \\ \sqrt[3]{27} &= \sqrt[3]{x^3} \\ \sqrt[3]{27} &= x \\ 3 &= x \end{aligned}$$

Check:

$$\begin{aligned} 27 &= 3^3 \\ 27 &= 27 \end{aligned}$$

The cube has side lengths of 3 in.

5. What positive value of x makes the following equation true: $x^2 = 64$? Explain.

$$\begin{aligned} x^2 &= 64 \\ \sqrt{x^2} &= \sqrt{64} \\ x &= \sqrt{64} \\ x &= 8 \end{aligned}$$

Check:

$$\begin{aligned} 8^2 &= 64 \\ 64 &= 64 \end{aligned}$$

To solve the equation, I need to find the positive value of x so that when it is squared, it is equal to 64. Therefore, I can take the square root of both sides of the equation. The square root of x^2, $\sqrt{x^2}$, is x because $x^2 = x \cdot x$. The square root of 64, $\sqrt{64}$, is 8 because $64 = 8 \cdot 8$. Therefore, $x = 8$.

6. What positive value of x makes the following equation true: $x^3 = 64$? Explain.

$$\begin{aligned} x^3 &= 64 \\ \sqrt[3]{x^3} &= \sqrt[3]{64} \\ x &= \sqrt[3]{64} \\ x &= 4 \end{aligned}$$

Check:

$$\begin{aligned} 4^3 &= 64 \\ 64 &= 64 \end{aligned}$$

To solve the equation, I need to find the positive value of x so that when it is cubed, it is equal to 64. Therefore, I can take the cube root of both sides of the equation. The cube root of x^3, $\sqrt[3]{x^3}$, is x because $x^3 = x \cdot x \cdot x$. The cube root of 64, $\sqrt[3]{64}$, is 4 because $64 = 4 \cdot 4 \cdot 4$. Therefore, $x = 4$.

7. **Find the positive value of x that makes the equation true: $x^2 = 256^{-1}$.**

$$x^2 = 256^{-1}$$
$$\sqrt{x^2} = \sqrt{256^{-1}}$$
$$x = \sqrt{256^{-1}}$$
$$x = \sqrt{\frac{1}{256}}$$
$$x = \frac{1}{16}$$
$$x = 16^{-1}$$

Check:

$$(16^{-1})^2 = 256^{-1}$$
$$16^{-2} = 256^{-1}$$
$$\frac{1}{16^2} = 256^{-1}$$
$$\frac{1}{256} = 256^{-1}$$
$$256^{-1} = 256^{-1}$$

8. **Find the positive value of x that makes the equation true: $x^3 = 343^{-1}$.**

$$x^3 = 343^{-1}$$
$$\sqrt[3]{x^3} = \sqrt[3]{343^{-1}}$$
$$x = \sqrt[3]{343^{-1}}$$
$$x = \sqrt[3]{\frac{1}{343}}$$
$$x = \frac{1}{7}$$
$$x = 7^{-1}$$

Check:

$$(7^{-1})^3 = 343^{-1}$$
$$7^{-3} = 343^{-1}$$
$$\frac{1}{7^3} = 343^{-1}$$
$$\frac{1}{343} = 343^{-1}$$
$$343^{-1} = 343^{-1}$$

MP.6

9. **Is 6 a solution to the equation $x^2 - 4 = 5x$? Explain why or why not.**

$$6^2 - 4 = 5(6)$$
$$36 - 4 = 30$$
$$32 \neq 30$$

No, 6 is not a solution to the equation $x^2 - 4 = 5x$. When the number is substituted into the equation and simplified, the left side of the equation and the right side of the equation are not equal; in other words, it is not a true number sentence. Since the number 6 does not satisfy the equation, then it is not a solution to the equation.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that the positive n^{th} root of a number exists and is unique.
- We know how to solve equations that contain exponents of 2 and 3; we must use square roots and cube roots.

Lesson Summary

The symbol $\sqrt[n]{}$ is called a *radical*. An equation that contains that symbol is referred to as a *radical equation*. So far, we have only worked with square roots ($n = 2$). Technically, we would denote a positive square root as $\sqrt[2]{}$, but it is understood that the symbol $\sqrt{}$ alone represents a positive square root.

When $n = 3$, then the symbol $\sqrt[3]{}$ is used to denote the cube root of a number. Since $x^3 = x \cdot x \cdot x$, then the cube root of x^3 is x; in other words, $\sqrt[3]{x^3} = x$.

The square or cube root of a positive number exists, and there can be only one positive square root or one cube root of the number.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 3: Existence and Uniqueness of Square Roots and Cube Roots

Exit Ticket

Find the positive value of x that makes each equation true. Check your solution.

1. $x^2 = 225$
   a. Explain the first step in solving this equation.
   b. Solve and check your solution.

2. $x^3 = 512$

3. $x^2 = 361^{-1}$

4. $x^3 = 1000^{-1}$

Exit Ticket Sample Solutions

Find the positive value of x that makes each equation true. Check your solution.

1. $x^2 = 225$

   a. **Explain the first step in solving this equation.**

   The first step is to take the square root of both sides of the equation.

   b. **Solve and check your solution.**

$$x^2 = 225$$
$$\sqrt{x^2} = \sqrt{225}$$
$$x = \sqrt{225}$$
$$x = 15$$

Check:

$$15^2 = 225$$
$$225 = 225$$

2. $x^3 = 512$

$$x^3 = 512$$
$$\sqrt[3]{x^3} = \sqrt[3]{512}$$
$$x = \sqrt[3]{512}$$
$$x = 8$$

Check:

$$8^3 = 512$$
$$512 = 512$$

3. $x^2 = 361^{-1}$

$$x^2 = 361^{-1}$$
$$\sqrt{x^2} = \sqrt{361^{-1}}$$
$$x = \sqrt{361^{-1}}$$
$$x = \sqrt{\frac{1}{361}}$$
$$x = \frac{1}{19}$$
$$x = 19^{-1}$$

Check:

$$(19^{-1})^2 = 361^{-1}$$
$$19^{-2} = 361^{-1}$$
$$\frac{1}{19^2} = 361^{-1}$$
$$\frac{1}{361} = 361^{-1}$$
$$361^{-1} = 361^{-1}$$

4. $x^3 = 1000^{-1}$

$$x^3 = 1000^{-1}$$
$$\sqrt[3]{x^3} = \sqrt[3]{1000^{-1}}$$
$$x = \sqrt[3]{1000^{-1}}$$
$$x = \sqrt[3]{\frac{1}{1000}}$$
$$x = \frac{1}{10}$$
$$x = 10^{-1}$$

Check:

$$(10^{-1})^3 = 1000^{-1}$$
$$10^{-3} = 1000^{-1}$$
$$\frac{1}{10^3} = 1000^{-1}$$
$$\frac{1}{1000} = 1000^{-1}$$
$$1000^{-1} = 1000^{-1}$$

Problem Set Sample Solutions

Find the positive value of x that makes each equation true. Check your solution.

1. **What positive value of x makes the following equation true: $x^2 = 289$? Explain.**

$$x^2 = 289$$
$$\sqrt{x^2} = \sqrt{289}$$
$$x = \sqrt{289}$$
$$x = 17$$

Check:

$$17^2 = 289$$
$$289 = 289$$

To solve the equation, I need to find the positive value of x so that when it is squared, it is equal to 289. Therefore, I can take the square root of both sides of the equation. The square root of x^2, $\sqrt{x^2}$, is x because $x^2 = x \cdot x$. The square root of 289, $\sqrt{289}$, is 17 because $289 = 17 \cdot 17$. Therefore, $x = 17$.

2. **A square-shaped park has an area of 400 ft^2. What are the dimensions of the park? Write and solve an equation.**

$$x^2 = 400$$
$$\sqrt{x^2} = \sqrt{400}$$
$$x = \sqrt{400}$$
$$x = 20$$

Check:

$$20^2 = 400$$
$$400 = 400$$

The square park is 20 ft. in length and 20 ft. in width.

3. **A cube has a volume of 64 in^3. What is the measure of one of its sides? Write and solve an equation.**

$$x^3 = 64$$
$$\sqrt[3]{x^3} = \sqrt[3]{64}$$
$$x = \sqrt[3]{64}$$
$$x = 4$$

Check:

$$4^3 = 64$$
$$64 = 64$$

The cube has a side length of 4 in.

4. **What positive value of x makes the following equation true: $125 = x^3$? Explain.**

$$125 = x^3$$
$$\sqrt[3]{125} = \sqrt[3]{x^3}$$
$$\sqrt[3]{125} = x$$
$$5 = x$$

Check:

$$125 = 5^3$$
$$125 = 125$$

To solve the equation, I need to find the positive value of x so that when it is cubed, it is equal to 125. Therefore, I can take the cube root of both sides of the equation. The cube root of x^3, $\sqrt[3]{x^3}$, is x because $x^3 = x \cdot x \cdot x$. The cube root of 125, $\sqrt[3]{125}$, is 5 because $125 = 5 \cdot 5 \cdot 5$. Therefore, $x = 5$.

5. **Find the positive value of x that makes the equation true: $x^2 = 441^{-1}$.**
   a. **Explain the first step in solving this equation.**

   The first step is to take the square root of both sides of the equation.

b. Solve and check your solution.

$$x^2 = 441^{-1}$$
$$\sqrt{x^2} = \sqrt{441^{-1}}$$
$$x = \sqrt{441^{-1}}$$
$$x = \sqrt{\frac{1}{441}}$$
$$x = \frac{1}{21}$$
$$x = 21^{-1}$$

Check:

$$(21^{-1})^2 = 441^{-1}$$
$$21^{-2} = 441^{-1}$$
$$\frac{1}{21^2} = 441^{-1}$$
$$\frac{1}{441} = 441^{-1}$$
$$441^{-1} = 441^{-1}$$

6. Find the positive value of x that makes the equation true: $x^3 = 125^{-1}$.

$$x^3 = 125^{-1}$$
$$\sqrt[3]{x^3} = \sqrt[3]{125^{-1}}$$
$$x = \sqrt[3]{125^{-1}}$$
$$x = \sqrt[3]{\frac{1}{125}}$$
$$x = \frac{1}{5}$$
$$x = 5^{-1}$$

Check:

$$(5^{-1})^3 = 125^{-1}$$
$$5^{-3} = 125^{-1}$$
$$\frac{1}{5^3} = 125^{-1}$$
$$\frac{1}{125} = 125^{-1}$$
$$125^{-1} = 125^{-1}$$

7. The area of a square is 196 in^2. What is the length of one side of the square? Write and solve an equation, and then check your solution.

Let x represent the length of one side of the square.

$$x^2 = 196$$
$$\sqrt{x^2} = \sqrt{196}$$
$$x = \sqrt{196}$$
$$x = 14$$

Check:

$$14^2 = 196$$
$$196 = 196$$

The length of one side of the square is 14 in.

8. The volume of a cube is 729 cm^3. What is the length of one side of the cube? Write and solve an equation, and then check your solution.

Let x represent the length of one side of the cube.

$$x^3 = 729$$
$$\sqrt[3]{x^3} = \sqrt[3]{729}$$
$$x = \sqrt[3]{729}$$
$$x = 9$$

Check:

$$9^3 = 729$$
$$729 = 729$$

The length of one side of the cube is 9 cm.

9. **What positive value of x would make the following equation true: $19 + x^2 = 68$?**

$$19 + x^2 = 68$$
$$19 - 19 + x^2 = 68 - 19$$
$$x^2 = 49$$
$$x = 7$$

The positive value for x that makes the equation true is 7.

Lesson 4: Simplifying Square Roots

Student Outcomes

- Students use factors of a number to simplify a square root.

Lesson Notes

This lesson is optional. In this lesson, students learn to simplify square roots by examining the factors of a number and looking specifically for perfect squares. Students must learn how to work with square roots in Grade 8 in preparation for their work in Algebra I and the quadratic formula. Though this lesson is optional, it is strongly recommended that students learn how to work with numbers in radical form in preparation for the work that they will do in Algebra I. Throughout the remaining lessons of this module, students will work with dimensions in the form of a simplified square root and learn to express answers as a simplified square root to increase their fluency in working with numbers in this form.

Classwork

Opening Exercise (5 minutes)

MP.8

Opening Exercise

a.
   i. What does $\sqrt{16}$ equal?
   ii. What does 4×4 equal?
   iii. Does $\sqrt{16} = \sqrt{4 \times 4}$?

b.
   i. What does $\sqrt{36}$ equal?
   ii. What does 6×6 equal?
   iii. Does $\sqrt{36} = \sqrt{6 \times 6}$?

c.
   i. What does $\sqrt{121}$ equal?
   ii. What does 11×11 equal?
   iii. Does $\sqrt{121} = \sqrt{11 \times 11}$?

d.
   i. What does $\sqrt{81}$ equal?
   ii. What does 9×9 equal?
   iii. Does $\sqrt{81} = \sqrt{9 \times 9}$?

e. What is another way to write $\sqrt{20}$?

$\sqrt{20} = \sqrt{4 \times 5}$

f. What is another way to write $\sqrt{28}$?

$\sqrt{28} = \sqrt{4 \times 7}$

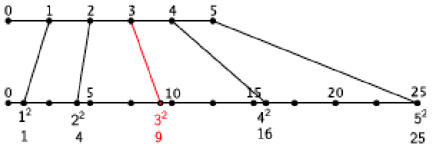
Discussion (7 minutes)

- We know from the last lesson that square roots can be simplified to a whole number when they are perfect squares. That is,

$$\sqrt{9} = \sqrt{3 \times 3} = \sqrt{3^2} = 3.$$

Scaffolding:

If necessary, remind students about numbers and their squares using the visual below.



- Given x^2 (x is a positive integer and x squared is a perfect square), it is easy to see that when $C = \sqrt{x^2}$ and $D = x$ that $C = D$, where C and D are positive numbers. In terms of the previous example, when $C = \sqrt{9} = \sqrt{3^2}$ and $D = 3$, then $3 = 3$.
- We can show that this is true even when we do not have perfect squares. All we need to show is that when C and D are positive numbers and n is a positive integer, that $C^n = D^n$. If we can show that $C^n = D^n$, then we know that $C = D$.

Ask students to explain why $C^n = D^n$ implies $C = D$. They should reference the definition of exponential notation that they learned in Module 1. For example, since $C^n = \underbrace{C \times C \times \cdots \times C}_{n \text{ times}}$ and $D^n = \underbrace{D \times D \times \cdots \times D}_{n \text{ times}}$, and we are given that $\underbrace{C \times C \times \cdots \times C}_{n \text{ times}} = \underbrace{D \times D \times \cdots \times D}_{n \text{ times}}$, then C must be the same number as D.

Scaffolding:

Students may know the term *product* in other contexts. For that reason, students may need to be reminded that it refers to multiplication in this context.

- Now, for the proof that the n^{th} root of a number can be expressed as a product of the n^{th} root of its factors:

  Let $C = \sqrt[n]{ab}$ and $D = \sqrt[n]{a} \times \sqrt[n]{b}$, where a and b are positive integers and n is a positive integer greater than or equal to 2. We want to show that $C^n = D^n$.

$$\begin{aligned} C^n &= \left(\sqrt[n]{ab}\right)^n \\ &= \underbrace{\sqrt[n]{ab} \times \sqrt[n]{ab} \times \cdots \times \sqrt[n]{ab}}_{n \text{ times}} \\ &= ab \end{aligned}$$

$$\begin{aligned} D^n &= \left(\sqrt[n]{a} \times \sqrt[n]{b}\right)^n \\ &= \underbrace{\left(\sqrt[n]{a} \times \sqrt[n]{b}\right) \times \left(\sqrt[n]{a} \times \sqrt[n]{b}\right) \times \cdots \times \left(\sqrt[n]{a} \times \sqrt[n]{b}\right)}_{n \text{ times}} \\ &= \underbrace{\sqrt[n]{a} \times \sqrt[n]{a} \times \cdots \times \sqrt[n]{a}}_{n \text{ times}} \times \underbrace{\sqrt[n]{b} \times \sqrt[n]{b} \times \cdots \times \sqrt[n]{b}}_{n \text{ times}} \\ &= ab \end{aligned}$$

- Since $C^n = D^n$ implies $C = D$, then $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$.
- Let's look again at some concrete numbers. What is $\sqrt{36}$?
  - $\sqrt{36} = 6$
- Now, consider the factors of 36. Specifically, consider those that are perfect squares. We want to rewrite $\sqrt{36}$ as a product of perfect squares. What will that be?
  - $\sqrt{36} = \sqrt{4 \times 9}$
- Based on what we just learned, we can write $\sqrt{36} = \sqrt{4 \times 9} = \sqrt{4} \times \sqrt{9}$. What does the last expression simplify to? How does it compare to our original statement that $\sqrt{36} = 6$?
  - $\sqrt{4} \times \sqrt{9} = 2 \times 3 = 6$*; the answers are the same, so* $\sqrt{36} = \sqrt{4} \times \sqrt{9}$.

- Simplify $\sqrt{64}$ two different ways. Explain your work to a partner.
    - $\sqrt{64} = \sqrt{8 \times 8} = \sqrt{8^2} = 8$; *the number* 64 *is a product of* 8 *multiplied by itself, which is the same as* 8^2. *Since the square root symbol asks for the number that when multiplied by itself is* 64, *then* $\sqrt{64} = 8$.
    - $\sqrt{64} = \sqrt{16 \times 4} = \sqrt{16} \times \sqrt{4} = \sqrt{4^2} \times \sqrt{2^2} = 4 \times 2 = 8$; *the number* 64 *is a product of* 16 *and* 4. *We can first rewrite* $\sqrt{64}$ *as a product of its factors,* $\sqrt{16 \times 4}$, *then as* $\sqrt{16} \times \sqrt{4}$. *Each of the numbers* 16 *and* 4 *are perfect squares that can be simplified as before, so* $\sqrt{16} \times \sqrt{4} = \sqrt{4^2} \times \sqrt{2^2} = 4 \times 2 = 8$. *Therefore,* $\sqrt{64} = 8$. *This means that* $\sqrt{64} = \sqrt{16} \times \sqrt{4}$.

Example 1 (4 minutes)

Example 1

Simplify the square root as much as possible.

$\sqrt{\mathbf{50}} =$

- Is the number 50 a perfect square? Explain.
    - *The number* 50 *is not a perfect square because there is no integer squared that equals* 50.
- Since 50 is not a perfect square, when we need to simplify $\sqrt{50}$, we write the factors of the number 50 looking specifically for those that are perfect squares. What are the factors of 50?
    - $50 = 2 \times 5^2$
- Since $50 = 2 \times 5^2$, then $\sqrt{50} = \sqrt{2 \times 5^2}$. We can rewrite $\sqrt{50}$ as a product of its factors:

$$\sqrt{50} = \sqrt{2} \times \sqrt{5^2}.$$

Obviously, 5^2 is a perfect square. Therefore, $\sqrt{5^2} = 5$, so $\sqrt{50} = 5 \times \sqrt{2} = 5\sqrt{2}$. Since $\sqrt{2}$ is not a perfect square, we will leave it as it is.

- The number $\sqrt{50}$ is said to be in its simplified form when all perfect square factors have been simplified. Therefore, $5\sqrt{2}$ is the simplified form of $\sqrt{50}$.
- Now that we know $\sqrt{50}$ can be expressed as a product of its factors, we also know that we can multiply expressions containing square roots. For example, if we are given $\sqrt{2} \times \sqrt{5^2}$, we can rewrite the expression as $\sqrt{2 \times 5^2} = \sqrt{50}$.

Example 2 (3 minutes)

Example 2

Simplify the square root as much as possible.

$\sqrt{\mathbf{28}} =$

- Is the number 28 a perfect square? Explain.
  - *The number* 28 *is not a perfect square because there is no integer squared that equals* 28*.*
- What are the factors of 28?
  - $28 = 2^2 \times 7$
- Since $28 = 2^2 \times 7$, then $\sqrt{28} = \sqrt{2^2 \times 7}$. We can rewrite $\sqrt{28}$ as a product of its factors:

  $$\sqrt{28} = \sqrt{2^2} \times \sqrt{7}.$$

  Obviously, 2^2 is a perfect square. Therefore, $\sqrt{2^2} = 2$, and $\sqrt{28} = 2 \times \sqrt{7} = 2\sqrt{7}$. Since $\sqrt{7}$ is not a perfect square, we will leave it as it is.
- The number $\sqrt{28}$ is said to be in its simplified form when all perfect square factors have been simplified. Therefore, $2\sqrt{7}$ is the simplified form of $\sqrt{28}$.

Exercises 1–4 (5 minutes)

Students complete Exercises 1–4 independently.

Exercises 1–4

Simplify the square roots as much as possible.

1. $\sqrt{18}$

$$\begin{aligned}\sqrt{18} &= \sqrt{2 \times 3^2}\\ &= \sqrt{2} \times \sqrt{3^2}\\ &= 3\sqrt{2}\end{aligned}$$

2. $\sqrt{44}$

$$\begin{aligned}\sqrt{44} &= \sqrt{2^2 \times 11}\\ &= \sqrt{2^2} \times \sqrt{11}\\ &= 2\sqrt{11}\end{aligned}$$

3. $\sqrt{169}$

$$\begin{aligned}\sqrt{169} &= \sqrt{13^2}\\ &= 13\end{aligned}$$

4. $\sqrt{75}$

$$\begin{aligned}\sqrt{75} &= \sqrt{3 \times 5^2}\\ &= \sqrt{3} \times \sqrt{5^2}\\ &= 5\sqrt{3}\end{aligned}$$

Example 3 (4 minutes)

Example 3

Simplify the square root as much as possible.

$\sqrt{128} =$

In this example, students may or may not recognize 128 as 64×2. The work below assumes that they do not. Consider showing students the solution below, as well as this alternative solution:

$$\sqrt{128} = \sqrt{64 \times 2} = \sqrt{64} \times \sqrt{2} = 8 \times \sqrt{2} = 8\sqrt{2}.$$

- Is the number 128 a perfect square? Explain.
    - *The number 128 is not a perfect square because there is no integer squared that equals 128.*
- What are the factors of 128?
    - $128 = 2^7$
- Since $128 = 2^7$, then $\sqrt{128} = \sqrt{2^7}$. We know that we can simplify perfect squares so we can rewrite 2^7 as $2^2 \times 2^2 \times 2^2 \times 2$ because of what we know about the laws of exponents. Then,

  $$\sqrt{128} = \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2}.$$

  Each 2^2 is a perfect square. Therefore, $\sqrt{128} = 2 \times 2 \times 2 \times \sqrt{2} = 8\sqrt{2}$.

Example 4 (4 minutes)

Example 4

Simplify the square root as much as possible.

$\sqrt{\mathbf{288}} =$

In this example, students may or may not recognize 288 as 144×2. The work below assumes that they do not. Consider showing students the solution below, as well as this alternative solution:

$$\sqrt{288} = \sqrt{(144 \times 2)} = \sqrt{144} \times \sqrt{2} = 12 \times \sqrt{2} = 12\sqrt{2}.$$

- Is the number 288 a perfect square? Explain.
    - *The number 288 is not a perfect square because there is no integer squared that equals 288.*
- What are the factors of 288?
    - $288 = 2^5 \times 3^2$
- Since $288 = 2^5 \times 3^2$, then $\sqrt{288} = \sqrt{2^5 \times 3^2}$. What do we do next?
    - *Use the laws of exponents to rewrite 2^5 as $2^2 \times 2^2 \times 2$.*
- Then, $\sqrt{288}$ is equivalent to
    - $\sqrt{288} = \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2} \times \sqrt{3^2}$.
- What does this simplify to?
    - $\sqrt{288} = \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2} \times \sqrt{3^2} = \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{3^2} \times \sqrt{2} = 2 \times 2 \times 3 \times \sqrt{2} = 12\sqrt{2}$

Exercises 5–8 (5 minutes)

Students work independently or in pairs to complete Exercises 5–8.

Exercises 5–8

5. **Simplify $\sqrt{108}$.**

$$\begin{aligned}\sqrt{108} &= \sqrt{2^2 \times 3^3}\\ &= \sqrt{2^2} \times \sqrt{3^2} \times \sqrt{3}\\ &= 2 \times 3\sqrt{3}\\ &= 6\sqrt{3}\end{aligned}$$

6. **Simplify $\sqrt{250}$.**

$$\begin{aligned}\sqrt{250} &= \sqrt{2 \times 5^3}\\ &= \sqrt{2} \times \sqrt{5^2} \times \sqrt{5}\\ &= 5\sqrt{2} \times \sqrt{5}\\ &= 5\sqrt{10}\end{aligned}$$

7. **Simplify $\sqrt{200}$.**

$$\begin{aligned}\sqrt{200} &= \sqrt{2^3 \times 5^2}\\ &= \sqrt{2^2} \times \sqrt{2} \times \sqrt{5^2}\\ &= 2 \times 5\sqrt{2}\\ &= 10\sqrt{2}\end{aligned}$$

8. **Simplify $\sqrt{504}$.**

$$\begin{aligned}\sqrt{504} &= \sqrt{2^3 \times 3^2 \times 7}\\ &= \sqrt{2^2} \times \sqrt{2} \times \sqrt{3^2} \times \sqrt{7}\\ &= 2 \times 3 \times \sqrt{2} \times \sqrt{7}\\ &= 6\sqrt{14}\end{aligned}$$

Scaffolding:

Some simpler problems are included here.

- Simplify $\sqrt{12}$.
    - $\sqrt{12} = \sqrt{2^2 \times 3}$
      $= \sqrt{2^2} \times \sqrt{3}$
      $= 2 \times \sqrt{3}$
      $= 2\sqrt{3}$
- Simplify $\sqrt{48}$.
    - $\sqrt{48} = \sqrt{2^4 \times 3}$
      $= \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{3}$
      $= 2 \times 2 \times \sqrt{3}$
      $= 4\sqrt{3}$
- Simplify $\sqrt{350}$.
    - $\sqrt{350} = \sqrt{5^2 \times 2 \times 7}$
      $= \sqrt{5^2} \times \sqrt{2} \times \sqrt{7}$
      $= 5 \times \sqrt{2} \times \sqrt{7}$
      $= 5\sqrt{14}$

Closing (3 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know how to simplify a square root by using the factors of a given number and then simplifying the perfect squares.

Lesson Summary

Square roots of non-perfect squares can be simplified by using the factors of the number. Any perfect square factors of a number can be simplified.

For example:

$$\begin{aligned}\sqrt{72} &= \sqrt{36 \times 2}\\ &= \sqrt{36} \times \sqrt{2}\\ &= \sqrt{6^2} \times \sqrt{2}\\ &= 6 \times \sqrt{2}\\ &= 6\sqrt{2}\end{aligned}$$

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 4: Simplifying Square Roots

Exit Ticket

Simplify the square roots as much as possible.

1. $\sqrt{24}$

2. $\sqrt{338}$

3. $\sqrt{196}$

4. $\sqrt{2420}$

Exit Ticket Sample Solutions

Simplify the square roots as much as possible.

1. $\sqrt{24}$

$$\begin{aligned}\sqrt{24} &= \sqrt{2^2 \times 6}\\ &= \sqrt{2^2} \times \sqrt{6}\\ &= 2\sqrt{6}\end{aligned}$$

2. $\sqrt{338}$

$$\begin{aligned}\sqrt{338} &= \sqrt{13^2 \times 2}\\ &= \sqrt{13^2} \times \sqrt{2}\\ &= 13\sqrt{2}\end{aligned}$$

3. $\sqrt{196}$

$$\begin{aligned}\sqrt{196} &= \sqrt{14^2}\\ &= 14\end{aligned}$$

4. $\sqrt{2,420}$

$$\begin{aligned}\sqrt{2420} &= \sqrt{2^2 \times 11^2 \times 5}\\ &= \sqrt{2^2} \times \sqrt{11^2} \times \sqrt{5}\\ &= 2 \times 11 \times \sqrt{5}\\ &= 22\sqrt{5}\end{aligned}$$

Problem Set Sample Solutions

Simplify each of the square roots in Problems 1–5 as much as possible.

1. $\sqrt{98}$

$$\begin{aligned}\sqrt{98} &= \sqrt{2 \times 7^2}\\ &= \sqrt{2} \times \sqrt{7^2}\\ &= 7\sqrt{2}\end{aligned}$$

2. $\sqrt{54}$

$$\begin{aligned}\sqrt{54} &= \sqrt{2 \times 3^3}\\ &= \sqrt{2} \times \sqrt{3} \times \sqrt{3^2}\\ &= 3\sqrt{6}\end{aligned}$$

3. $\sqrt{144}$

$$\begin{aligned}\sqrt{144} &= \sqrt{12^2}\\ &= 12\end{aligned}$$

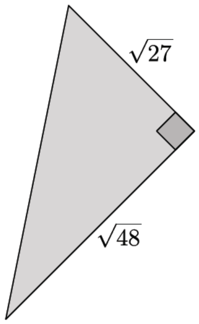
4. $\sqrt{512}$

$$\begin{aligned}\sqrt{512} &= \sqrt{2^9}\\ &= \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2}\\ &= 2 \times 2 \times 2 \times 2\sqrt{2}\\ &= 16\sqrt{2}\end{aligned}$$

5. $\sqrt{756}$

$$\begin{aligned}\sqrt{756} &= \sqrt{2^2 \times 3^3 \times 7}\\ &= \sqrt{2^2} \times \sqrt{3^2} \times \sqrt{3} \times \sqrt{7}\\ &= 2 \times 3 \times \sqrt{21}\\ &= 6\sqrt{21}\end{aligned}$$

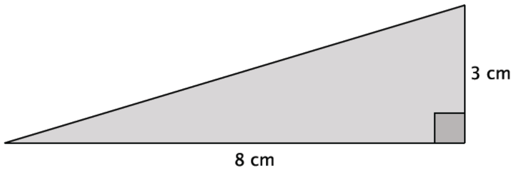
6. **What is the length of the unknown side of the right triangle? Simplify your answer, if possible.**



Let c represent the length of the hypotenuse.

$$\left(\sqrt{27}\right)^2 + \left(\sqrt{48}\right)^2 = c^2$$
$$27 + 48 = c^2$$
$$75 = c^2$$
$$\sqrt{75} = \sqrt{c^2}$$
$$\sqrt{5^2} \times \sqrt{3} = c$$
$$5\sqrt{3} = c$$

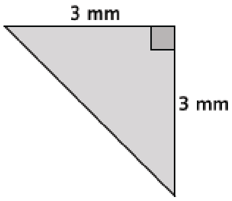
7. **What is the length of the unknown side of the right triangle? Simplify your answer, if possible.**



Let c represent the length of the hypotenuse.

$$3^2 + 8^2 = c^2$$
$$9 + 64 = c^2$$
$$73 = c^2$$
$$\sqrt{73} = \sqrt{c^2}$$
$$\sqrt{73} = c$$

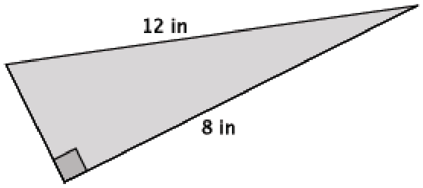
8. **What is the length of the unknown side of the right triangle? Simplify your answer, if possible.**



Let c represent the length of the hypotenuse.

$$3^2 + 3^2 = c^2$$
$$9 + 9 = c^2$$
$$18 = c^2$$
$$\sqrt{18} = \sqrt{c^2}$$
$$\sqrt{18} = c$$
$$\sqrt{3^2} \times \sqrt{2} = c$$
$$3\sqrt{2} = c$$

9. **What is the length of the unknown side of the right triangle? Simplify your answer, if possible.**



Let x represent the unknown length.

$$x^2 + 8^2 = 12^2$$
$$x^2 + 64 = 144$$
$$x^2 + 64 - 64 = 144 - 64$$
$$x^2 = 80$$
$$\sqrt{x^2} = \sqrt{80}$$
$$x = \sqrt{80}$$
$$x = \sqrt{2^4 \times 5}$$
$$x = \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{5}$$
$$x = 2 \times 2\sqrt{5}$$
$$x = 4\sqrt{5}$$

10. **Josue simplified $\sqrt{450}$ as $15\sqrt{2}$. Is he correct? Explain why or why not.**

$$\begin{aligned}\sqrt{450} &= \sqrt{2 \times 3^2 \times 5^2}\\ &= \sqrt{2} \times \sqrt{3^2} \times \sqrt{5^2}\\ &= 3 \times 5 \times \sqrt{2}\\ &= 15\sqrt{2}\end{aligned}$$

Yes, Josue is correct because the number $450 = 2 \times 3^2 \times 5^2$. The factors that are perfect squares simplify to 15 leaving just the factor of 2 that cannot be simplified. Therefore, $\sqrt{450} = 15\sqrt{2}$.

11. **Tiah was absent from school the day that you learned how to simplify a square root. Using $\sqrt{360}$, write Tiah an explanation for simplifying square roots.**

To simplify $\sqrt{360}$, first write the factors of 360. The number $360 = 2^3 \times 3^2 \times 5$. Now, we can use the factors to write $\sqrt{360} = \sqrt{2^3 \times 3^2 \times 5}$, which can then be expressed as $\sqrt{360} = \sqrt{2^3} \times \sqrt{3^2} \times \sqrt{5}$. Because we want to simplify square roots, we can rewrite the factor $\sqrt{2^3}$ as $\sqrt{2^2} \times \sqrt{2}$ because of the laws of exponents. Now, we have

$$\sqrt{360} = \sqrt{2^2} \times \sqrt{2} \times \sqrt{3^2} \times \sqrt{5}.$$

Each perfect square can be simplified as follows:

$$\begin{aligned}\sqrt{360} &= 2 \times \sqrt{2} \times 3 \times \sqrt{5}\\ &= 2 \times 3 \times \sqrt{2} \times \sqrt{5}\\ &= 6\sqrt{10}.\end{aligned}$$

The simplified version of $\sqrt{360} = 6\sqrt{10}$.

Lesson 5: Solving Radical Equations

Student Outcomes

- Students find the positive solutions for equations of the form $x^2 = p$ and $x^3 = p$.

Classwork

Discussion (15 minutes)

- Just recently, we began solving equations that required us to find the square root or cube root of a number. All of those equations were in the form of $x^2 = p$ or $x^3 = p$, where p was a positive rational number.

Example 1

Example 1

$$x^3 + 9x = \frac{1}{2}(18x + 54)$$

MP.1

- Now that we know a little more about square roots and cube roots, we can begin solving nonlinear equations like $x^3 + 9x = \frac{1}{2}(18x + 54)$. Transform the equation using our properties of equality until you can determine the positive value of x that makes the equation true.

Challenge students to solve the equation independently or in pairs. Have students share their strategy for solving the equation. Ask them to explain each step. Although we are asking students to find the positive value of x that makes each equation true, we have included in the examples both the positive and the negative values of x for your reference if you choose to use them.

- *Sample response:*

$$\begin{aligned} x^3 + 9x &= \frac{1}{2}(18x + 54) \\ x^3 + 9x &= 9x + 27 \\ x^3 + 9x - 9x &= 9x - 9x + 27 \\ x^3 &= 27 \\ \sqrt[3]{x^3} &= \sqrt[3]{27} \\ x &= \sqrt[3]{3^3} \\ x &= 3 \end{aligned}$$

Scaffolding:

Consider using a simpler version of the equation (line 2, for example):

$$x^3 + 9x = 9x + 27.$$

- Now, we verify our solution is correct.

$$\begin{aligned} 3^3 + 9(3) &= \frac{1}{2}(18(3) + 54) \\ 27 + 27 &= \frac{1}{2}(56 + 54) \\ 54 &= \frac{1}{2}(108) \\ 54 &= 54 \end{aligned}$$

- Since the left side is the same as the right side, our solution is correct.

Example 2

Example 2

$$\boldsymbol{x(x-3) - 51 = -3x + 13}$$

- Let's look at another nonlinear equation. Find the positive value of x that makes the equation true: $x(x-3) - 51 = -3x + 13$.

Provide students with time to solve the equation independently or in pairs. Have students share their strategy for solving the equation. Ask them to explain each step.

- *Sample response:*

$$\begin{aligned} x(x-3) - 51 &= -3x + 13 \\ x^2 - 3x - 51 &= -3x + 13 \\ x^2 - 3x + 3x - 51 &= -3x + 3x + 13 \\ x^2 - 51 &= 13 \\ x^2 - 51 + 51 &= 13 + 51 \\ x^2 &= 64 \\ \sqrt{x^2} &= \pm\sqrt{64} \\ x &= \pm\sqrt{64} \\ x &= \pm 8 \end{aligned}$$

- Now we verify our solution is correct.

Provide students time to check their work.

- Let $x = 8$.

$$\begin{aligned} 8(8-3) - 51 &= -3(8) + 13 \\ 8(5) - 51 &= -24 + 13 \\ 40 - 51 &= -11 \\ -11 &= -11 \end{aligned}$$

Let $x = -8$.

$$\begin{aligned} -8(-8-3) - 51 &= -3(-8) + 13 \\ -8(-11) - 51 &= 24 + 13 \\ 88 - 51 &= 37 \\ 37 &= 37 \end{aligned}$$

- Now it is clear that the left side is exactly the same as the right side, and our solution is correct.

Exercises 1–7 (20 minutes)

Students complete Exercises 1–7 independently or in pairs. Although we are asking students to find the positive value of x that makes each equation true, we have included in the exercises both the positive and the negative values of x for your reference if you choose to use them.

Exercises 1–7

Find the positive value of x that makes each equation true, and then verify your solution is correct.

1.

a. **Solve $x^2 - 14 = 5x + 67 - 5x$.**

$$\begin{aligned} x^2 - 14 &= 5x + 67 - 5x \\ x^2 - 14 &= 67 \\ x^2 - 14 + 14 &= 67 + 14 \\ x^2 &= 81 \\ \sqrt{x^2} &= \pm\sqrt{81} \\ x &= \pm\sqrt{81} \\ x &= \pm 9 \end{aligned}$$

Check:

$$\begin{aligned} 9^2 - 14 &= 5(9) + 67 - 5(9) \\ 81 - 14 &= 45 + 67 - 45 \\ 67 &= 67 \end{aligned}$$

$$\begin{aligned} (-9)^2 - 14 &= 5(-9) + 67 - 5(-9) \\ 81 - 14 &= -45 + 67 + 45 \\ 67 &= 67 \end{aligned}$$

b. **Explain how you solved the equation.**

To solve the equation, I had to first use the properties of equality to transform the equation into the form of $x^2 = 81$. Then, I had to take the square root of both sides of the equation to determine that $x = 9$ since the number x is being squared.

2. **Solve and simplify: $x(x - 1) = 121 - x$**

$$\begin{aligned} x(x - 1) &= 121 - x \\ x^2 - x &= 121 - x \\ x^2 - x + x &= 121 - x + x \\ x^2 &= 121 \\ \sqrt{x^2} &= \pm\sqrt{121} \\ x &= \pm\sqrt{121} \\ x &= \pm 11 \end{aligned}$$

Check:

$$\begin{aligned} 11(11 - 1) &= 121 - 11 \\ 11(10) &= 110 \\ 110 &= 110 \end{aligned}$$

$$\begin{aligned} -11(-11 - 1) &= 121 - (-11) \\ -11(-12) &= 121 + 11 \\ 132 &= 132 \end{aligned}$$

3. **A square has a side length of $3x$ and an area of 324 in^2. What is the value of x?**

$$\begin{aligned} (3x)^2 &= 324 \\ 3^2x^2 &= 324 \\ 9x^2 &= 324 \\ \frac{9x^2}{9} &= \frac{324}{9} \\ x^2 &= 36 \\ \sqrt{x^2} &= \sqrt{36} \\ x &= 6 \end{aligned}$$

Check:

$$\begin{aligned} (3 \times 6)^2 &= 324 \\ 18^2 &= 324 \\ 324 &= 324 \end{aligned}$$

A negative number would not make sense as a length, so $x = 6$.

4. $-3x^3 + 14 = -67$

$$\begin{aligned}
-3x^3 + 14 &= -67\\
-3x^3 + 14 - 14 &= -67 - 14\\
-3x^3 &= -81\\
\frac{-3x^3}{-3} &= \frac{-81}{-3}\\
x^3 &= 27\\
\sqrt[3]{x^3} &= \sqrt[3]{27}\\
x &= 3
\end{aligned}$$

Check:

$$\begin{aligned}
-3(3)^3 + 14 &= -67\\
-3(27) + 14 &= -67\\
-81 + 14 &= -67\\
-67 &= -67
\end{aligned}$$

5. $x(x+4) - 3 = 4(x + 19.5)$

$$\begin{aligned}
x(x+4) - 3 &= 4(x + 19.5)\\
x^2 + 4x - 3 &= 4x + 78\\
x^2 + 4x - 4x - 3 &= 4x - 4x + 78\\
x^2 - 3 &= 78\\
x^2 - 3 + 3 &= 78 + 3\\
x^2 &= 81\\
\sqrt{x^2} &= \pm\sqrt{81}\\
x &= \pm 9
\end{aligned}$$

Check:

$$\begin{aligned}
9(9+4) - 3 &= 4(9 + 19.5)\\
9(13) - 3 &= 4(28.5)\\
117 - 3 &= 114\\
114 &= 114
\end{aligned}$$

$$\begin{aligned}
-9(-9+4) - 3 &= 4(-9 + 19.5)\\
-9(-5) - 3 &= 4(10.5)\\
45 - 3 &= 42\\
42 &= 42
\end{aligned}$$

6. $216 + x = x(x^2 - 5) + 6x$

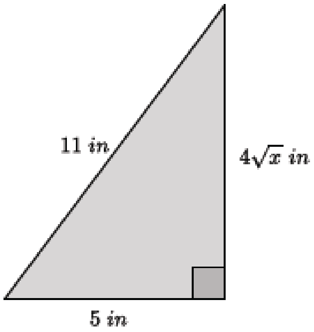
$$\begin{aligned}
216 + x &= x(x^2 - 5) + 6x\\
216 + x &= x^3 - 5x + 6x\\
216 + x &= x^3 + x\\
216 + x - x &= x^3 + x - x\\
216 &= x^3\\
\sqrt[3]{216} &= \sqrt[3]{x^3}\\
6 &= x
\end{aligned}$$

Check:

$$\begin{aligned}
216 + 6 &= 6(6^2 - 5) + 6(6)\\
222 &= 6(31) + 36\\
222 &= 186 + 36\\
222 &= 222
\end{aligned}$$

7.

a. **What are we trying to determine in the diagram below?**



We need to determine the value of x so that its square root, multiplied by 4, satisfies the equation $5^2 + \left(4\sqrt{x}\right)^2 = 11^2$.

b. Determine the value of x, and check your answer.

$$\begin{aligned} 5^2 + \left(4\sqrt{x}\right)^2 &= 11^2 \\ 25 + 4^2\left(\sqrt{x}\right)^2 &= 121 \\ 25 - 25 + 4^2\left(\sqrt{x}\right)^2 &= 121 - 25 \\ 16x &= 96 \\ \frac{16x}{16} &= \frac{96}{16} \\ x &= 6 \end{aligned}$$

Check:

$$\begin{aligned} 5^2 + \left(4\sqrt{6}\right)^2 &= 11^2 \\ 25 + 16(6) &= 121 \\ 25 + 96 &= 121 \\ 121 &= 121 \end{aligned}$$

The value of x is 6 in.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know how to solve equations with squared and cubed variables and verify that our solutions are correct.

Lesson Summary

Equations that contain variables that are squared or cubed can be solved using the properties of equality and the definition of square and cube roots.

Simplify an equation until it is in the form of $x^2 = p$ or $x^3 = p$ where p is a positive rational number; then, take the square or cube root to determine the positive value of x.

Example:

Solve for x.

$$\begin{aligned} \frac{1}{2}(2x^2 + 10) &= 30 \\ x^2 + 5 &= 30 \\ x^2 + 5 - 5 &= 30 - 5 \\ x^2 &= 25 \\ \sqrt{x^2} &= \sqrt{25} \\ x &= 5 \end{aligned}$$

Check:

$$\begin{aligned} \frac{1}{2}(2(5)^2 + 10) &= 30 \\ \frac{1}{2}(2(25) + 10) &= 30 \\ \frac{1}{2}(50 + 10) &= 30 \\ \frac{1}{2}(60) &= 30 \\ 30 &= 30 \end{aligned}$$

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 5: Solving Radical Equations

Exit Ticket

1. Find the positive value of x that makes the equation true, and then verify your solution is correct.

$$x^2 + 4x = 4(x + 16)$$

2. Find the positive value of x that makes the equation true, and then verify your solution is correct.

$$(4x)^3 = 1728$$

Exit Ticket Sample Solutions

1. **Find the positive value of x that makes the equation true, and then verify your solution is correct.**

$$x^2 + 4x = 4(x + 16)$$

$$\begin{aligned} x^2 + 4x &= 4(x+16) \\ x^2 + 4x &= 4x + 64 \\ x^2 + 4x - 4x &= 4x - 4x + 64 \\ x^2 &= 64 \\ \sqrt{x^2} &= \sqrt{64} \\ x &= 8 \end{aligned}$$

Check:

$$\begin{aligned} 8^2 + 4(8) &= 4(8+16) \\ 64 + 32 &= 4(24) \\ 96 &= 96 \end{aligned}$$

2. **Find the positive value of x that makes the equation true, and then verify your solution is correct.**

$$(4x)^3 = 1728$$

$$\begin{aligned} (4x)^3 &= 1728 \\ 64x^3 &= 1728 \\ \frac{1}{64}(64x^3) &= (1728)\frac{1}{64} \\ x^3 &= 27 \\ \sqrt[3]{x^3} &= \sqrt[3]{27} \\ x &= 3 \end{aligned}$$

Check:

$$\begin{aligned} (4 \times 3)^3 &= 1728 \\ 12^3 &= 1728 \\ 1728 &= 1728 \end{aligned}$$

Problem Set Sample Solutions

Find the positive value of x that makes each equation true, and then verify your solution is correct.

1. $x^2(x + 7) = \frac{1}{2}(14x^2 + 16)$

$$\begin{aligned} x^2(x+7) &= \frac{1}{2}(14x^2 + 16) \\ x^3 + 7x^2 &= 7x^2 + 8 \\ x^3 + 7x^2 - 7x^2 &= 7x^2 - 7x^2 + 8 \\ x^3 &= 8 \\ \sqrt[3]{x^3} &= \sqrt[3]{8} \\ x &= 2 \end{aligned}$$

Check:

$$\begin{aligned} 2^2(2+7) &= \frac{1}{2}(14(2^2) + 16) \\ 4(9) &= \frac{1}{2}(56 + 16) \\ 36 &= \frac{1}{2}(72) \\ 36 &= 36 \end{aligned}$$

2. $x^3 = 1331^{-1}$

$$x^3 = 1331^{-1}$$
$$\sqrt[3]{x^3} = \sqrt[3]{1331^{-1}}$$
$$x = \sqrt[3]{\frac{1}{1331}}$$
$$x = \sqrt[3]{\frac{1}{11^3}}$$
$$x = \frac{1}{11}$$

Check:

$$\left(\frac{1}{11}\right)^3 = 1331^{-1}$$
$$\frac{1}{11^3} = 1331^{-1}$$
$$\frac{1}{1331} = 1331^{-1}$$
$$1331^{-1} = 1331^{-1}$$

3. Determine the positive value of x that makes the equation true, and then explain how you solved the equation.

$$\frac{x^9}{x^7} - 49 = 0$$

$$\frac{x^9}{x^7} - 49 = 0$$
$$x^2 - 49 = 0$$
$$x^2 - 49 + 49 = 0 + 49$$
$$x^2 = 49$$
$$\sqrt{x^2} = \sqrt{49}$$
$$x = 7$$

Check:

$$7^2 - 49 = 0$$
$$49 - 49 = 0$$
$$0 = 0$$

To solve the equation, I first had to simplify the expression $\frac{x^9}{x^7}$ *to* x^2*. Next, I used the properties of equality to transform the equation into* $x^2 = 49$*. Finally, I had to take the square root of both sides of the equation to solve for* x*.*

4. Determine the positive value of x that makes the equation true.

$$(8x)^2 = 1$$

$$(8x)^2 = 1$$
$$64x^2 = 1$$
$$\sqrt{64x^2} = \sqrt{1}$$
$$8x = 1$$
$$\frac{8x}{8} = \frac{1}{8}$$
$$x = \frac{1}{8}$$

Check:

$$\left(8\left(\frac{1}{8}\right)\right)^2 = 1$$
$$1^2 = 1$$
$$1 = 1$$

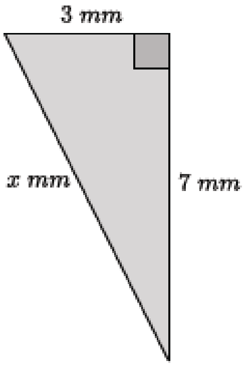
5. $\left(9\sqrt{x}\right)^2 - 43x = 76$

$$\begin{aligned} \left(9\sqrt{x}\right)^2 - 43x &= 76 \\ 9^2\left(\sqrt{x}\right)^2 - 43x &= 76 \\ 81x - 43x &= 76 \\ 38x &= 76 \\ \frac{38x}{38} &= \frac{76}{38} \\ x &= 2 \end{aligned}$$

Check:

$$\begin{aligned} (9(\sqrt{2}))^2 - 43(2) &= 76 \\ 9^2\left(\sqrt{2}\right)^2 - 86 &= 76 \\ 81(2) - 86 &= 76 \\ 162 - 86 &= 76 \\ 76 &= 76 \end{aligned}$$

6. **Determine the length of the hypotenuse of the right triangle below.**



$$\begin{aligned} 3^2 + 7^2 &= x^2 \\ 9 + 49 &= x^2 \\ 58 &= x^2 \\ \pm\sqrt{58} &= \sqrt{x^2} \\ \pm\sqrt{58} &= x \end{aligned}$$

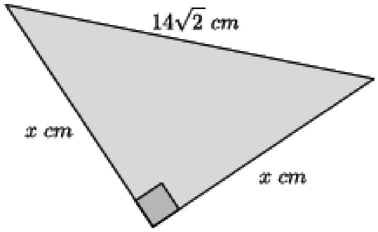
$$\sqrt{58} = x$$

A negative number would not make sense as a length, so $x = \sqrt{58}$ **mm.**

Check:

$$\begin{aligned} 3^2 + 7^2 &= \sqrt{58}^2 \\ 9 + 49 &= 58 \\ 58 &= 58 \end{aligned}$$

7. **Determine the length of the legs in the right triangle below.**



$$\begin{aligned} x^2 + x^2 &= \left(14\sqrt{2}\right)^2 \\ 2x^2 &= 14^2\left(\sqrt{2}\right)^2 \\ 2x^2 &= 196(2) \\ \frac{2x^2}{2} &= \frac{196(2)}{2} \\ x^2 &= 196 \\ \sqrt{x^2} &= \pm\sqrt{196} \\ x &= \pm\sqrt{14^2} \\ x &= \pm 14 \\ x &= 14 \end{aligned}$$

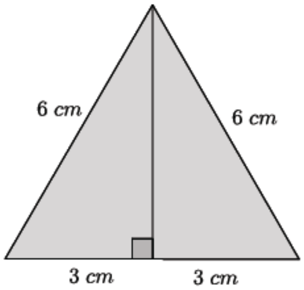
Check:

$$\begin{aligned} 14^2 + 14^2 &= \left(14\sqrt{2}\right)^2 \\ 196 + 196 &= 14^2\left(\sqrt{2}\right)^2 \\ 392 &= 196(2) \\ 392 &= 392 \end{aligned}$$

A negative number would not make sense as a length, so $x = 14$ cm.

8. **An equilateral triangle has side lengths of 6 cm. What is the height of the triangle? What is the area of the triangle?**

Note: This problem has two solutions, one with a simplified root and one without. Choose the appropriate solution for your classes based on how much simplifying you have taught them.



Let h represent the height of the triangle.

$$\begin{aligned} 3^2 + h^2 &= 6^2 \\ 9 + h^2 &= 36 \\ 9 - 9 + h^2 &= 36 - 9 \\ h^2 &= 27 \\ \sqrt{h^2} &= \sqrt{27} \\ h &= \sqrt{27} \\ h &= \sqrt{3^3} \\ h &= \sqrt{3^2} \times \sqrt{3} \\ h &= 3\sqrt{3} \end{aligned}$$

Let A represent the area of the triangle.

$$\begin{aligned} A &= \frac{6\left(3\sqrt{3}\right)}{2} \\ A &= 3\left(3\sqrt{3}\right) \\ A &= 9\sqrt{3} \end{aligned}$$

The height of the triangle is $3\sqrt{3}$ cm, and the area is $9\sqrt{3}$ cm^2.

The height of the triangle is $\sqrt{27}$cm, and the area is $3\sqrt{27}$ cm^2.

9. **Challenge: Find the positive value of x that makes the equation true.**

$$\left(\frac{1}{2}x\right)^2 - 3x = 7x + 8 - 10x$$

$$\begin{aligned}
\left(\frac{1}{2}x\right)^2 - 3x &= 7x + 8 - 10x \\
\frac{1}{4}x^2 - 3x &= -3x + 8 \\
\frac{1}{4}x^2 - 3x + 3x &= -3x + 3x + 8 \\
\frac{1}{4}x^2 &= 8 \\
4\left(\frac{1}{4}\right)x^2 &= 8(4) \\
x^2 &= 32 \\
\sqrt{x^2} &= \sqrt{32} \\
x &= \sqrt{2^5} \\
x &= \sqrt{2^2} \times \sqrt{2^2} \times \sqrt{2} \\
x &= 4\sqrt{2}
\end{aligned}$$

Check:

$$\begin{aligned}
\left(\frac{1}{2}(4\sqrt{2})\right)^2 - 3(4\sqrt{2}) &= 7(4\sqrt{2}) + 8 - 10(4\sqrt{2}) \\
\frac{1}{4}(16)(2) - 3(4\sqrt{2}) &= 7(4\sqrt{2}) - 10(4\sqrt{2}) + 8 \\
\frac{32}{4} - 3(4\sqrt{2}) &= 7(4\sqrt{2}) - 10(4\sqrt{2}) + 8 \\
8 - 3(4\sqrt{2}) &= (7 - 10)(4\sqrt{2}) + 8 \\
8 - 3(4\sqrt{2}) &= -3(4\sqrt{2}) + 8 \\
8 - 8 - 3(4\sqrt{2}) &= -3(4\sqrt{2}) + 8 - 8 \\
-3(4\sqrt{2}) &= -3(4\sqrt{2})
\end{aligned}$$

10. **Challenge: Find the positive value of x that makes the equation true.**

$$11x + x(x - 4) = 7(x + 9)$$

$$\begin{aligned}
11x + x(x - 4) &= 7(x + 9) \\
11x + x^2 - 4x &= 7x + 63 \\
7x + x^2 &= 7x + 63 \\
7x - 7x + x^2 &= 7x - 7x + 63 \\
x^2 &= 63 \\
\sqrt{x^2} &= \sqrt{63} \\
x &= \sqrt{3^2 \times 7} \\
x &= \sqrt{3^2} \times \sqrt{7} \\
x &= 3\sqrt{7}
\end{aligned}$$

Check:

$$\begin{aligned}
11(3\sqrt{7}) + 3\sqrt{7}(3\sqrt{7} - 4) &= 7(3\sqrt{7} + 9) \\
33\sqrt{7} + 3^2(\sqrt{7})^2 - 4(3\sqrt{7}) &= 21\sqrt{7} + 63 \\
33\sqrt{7} - 4(3\sqrt{7}) + 9(7) &= 21\sqrt{7} + 63 \\
33\sqrt{7} - 12\sqrt{7} + 63 &= 21\sqrt{7} + 63 \\
(33 - 12)\sqrt{7} + 63 &= 21\sqrt{7} + 63 \\
21\sqrt{7} + 63 &= 21\sqrt{7} + 63 \\
21\sqrt{7} + 63 - 63 &= 21\sqrt{7} + 63 - 63 \\
21\sqrt{7} &= 21\sqrt{7}
\end{aligned}$$

Topic B:

Decimal Expansions of Numbers

8.NS.A.1, 8.NS.A.2, 8.EE.A.2

Focus Standards:	8.NS.A.1	Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal expansion repeats eventually, and convert a decimal expansion which repeats eventually into a rational number.
	8.NS.A.2	Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of* $\sqrt{2}$, *show that* $\sqrt{2}$ *is between* 1 *and* 2, *then between* 1.4 *and* 1.5, *and explain how to continue to get better approximations.*
	8.EE.A.2	Use square root and cube root symbols to represent solutions to equations of the form $x^2 = p$ and $x^3 = p$, where p is a positive rational number. Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.
Instructional Days:	9	
Lesson 6:	Finite and Infinite Decimals (P)[1]	
Lesson 7:	Infinite Decimals (S)	
Lesson 8:	The Long Division Algorithm (E)	
Lesson 9:	Decimal Expansions of Fractions, Part 1 (P)	
Lesson 10:	Converting Repeating Decimals to Fractions (P)	
Lesson 11:	The Decimal Expansion of Some Irrational Numbers (S)	
Lesson 12:	Decimal Expansions of Fractions, Part 2 (S)	
Lesson 13:	Comparing Irrational Numbers (E)	
Lesson 14:	Decimal Expansion of π (S)	

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[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Throughout this topic, the terms *expanded form of a decimal* and *decimal expansion* are used. The expanded form of a decimal refers to the value of a number written as a sum. For example, the expanded form of the decimal 0.125 is $\frac{1}{10}+\frac{2}{10^2}+\frac{5}{10^3}$, which is closely related to the notion of expanded form used at the elementary level. When students are asked to determine the decimal expansion of a number such as $\sqrt{2}$, we expect them to write the number in decimal form. For example, the decimal expansion of $\sqrt{2}$ begins with 1.4142. The examination of the decimal expansion leads to an understanding of irrational numbers. Numbers with decimal expansions that are infinite (i.e., non-terminating) and that do not have a repeating block are called *irrational numbers*. Numbers with finite (i.e., terminating) decimal expansions, as well as those numbers that are infinite with repeating blocks, are called *rational numbers*. Students spend significant time engaging with finite and infinite decimals before the notion of an irrational number is introduced in Lesson 11.

In Lesson 6, students learn that every number has a decimal expansion that is finite or infinite. Finite and infinite decimals are defined, and students learn a strategy for writing a fraction as a finite decimal that focuses on the denominator and its factors. That is, a fraction can be written as a finite decimal if the denominator is a product of twos, a product of fives, or a product of twos and fives. In Lesson 7, students learn that numbers that cannot be expressed as finite decimals are infinite decimals. Students write the expanded form of infinite decimals and show on the number line their decimal representation in terms of intervals of tenths, hundredths, thousandths, and so on. This work with infinite decimals prepares students for understanding how to approximate the decimal expansion of an irrational number. In Lesson 8, students use the long division algorithm to determine the decimal form of a number and can relate the work of the algorithm to why digits in a decimal expansion repeat. It is in these first few lessons of Topic B that students recognize that rational numbers have a decimal expansion that repeats eventually, either in zeros or in a repeating block of digits. The discussion of infinite decimals continues with Lesson 9 where students learn how to use what they know about powers of 10 and equivalent fractions to make sense of why the long division algorithm can be used to convert a fraction to a decimal. Students know that multiplying the numerator and denominator of a fraction by a power of 10 is similar to putting zeros after the decimal point when doing long division.

In Lesson 10, students learn that a number with a decimal expansion that repeats can be expressed as a fraction. Students learn a strategy for writing repeating decimals as fractions that relies on their knowledge of multiplying by powers of 10 and solving linear equations. Lesson 11 introduces students to the method of rational approximation using a series of rational numbers to get closer and closer to a given number. Students write the approximate decimal expansion of irrational numbers in Lesson 11, and it is in this lesson that irrational numbers are defined as numbers that are not equal to rational numbers. Students realize that irrational numbers are different because they have infinite decimal expansions that do not repeat. Therefore, irrational numbers are those that are not equal to rational numbers. Rational approximation is used again in Lesson 12 to verify the decimal expansions of rational numbers. Students then compare the method of rational approximation to long division. In Lesson 13, students compare the value of rational and irrational numbers. Students use the method of rational approximation to determine the decimal expansion of an irrational number. Then, they compare that value to the decimal expansion of rational numbers in the form of a fraction, decimal, perfect square, or perfect cube. Students can now place irrational numbers on a number line with more accuracy than they did in Lesson 2. In Lesson 14, students approximate π using the area of a quarter circle that is drawn on grid paper. Students estimate the area of the quarter circle using inner and outer boundaries. As with the method of rational approximation, students continue to refine their estimates of the area, which improves their estimate of the value of π. Students then determine the approximate values of expressions involving π.

Lesson 6: Finite and Infinite Decimals

Student Outcomes

- Students know that every number has a decimal expansion (i.e., is equal to a finite or an infinite decimal).
- Students know that when a fraction has a denominator that is the product of 2's and/or 5's, it has a finite decimal expansion because the fraction can then be written in an equivalent form with a denominator that is a power of 10.

Lesson Notes

The terms *expanded form of a decimal* and *decimal expansion* are used throughout this topic. The expanded form of a decimal refers to the value of a number written as a sum. For example, the expanded form of the decimal 0.125 is $\frac{1}{10}+\frac{2}{10^2}+\frac{5}{10^3}$. When students are asked to determine the decimal expansion of a number like $\sqrt{2}$, we expect them to write the decimal value of the number. For example, the decimal expansion of $\sqrt{2}$ is approximately 1.4142. The examination of the decimal expansion leads to an understanding of irrational numbers. Numbers with decimal expansions that are infinite (i.e., non-terminating) and do not have a repeat block are called *irrational numbers*. Numbers with decimal expansions that are finite (i.e., terminating), as well as those numbers that are infinite with repeat blocks, are called *rational numbers*. Students will be exposed to the concepts of finite and infinite decimals here; however, the concept of irrational numbers will not be formally introduced until Lesson 11.

Classwork

Opening Exercise (7 minutes)

Provide students time to work, and then share their responses to part (e) with the class.

Opening Exercise

a. **Use long division to determine the decimal expansion of $\frac{54}{20}$.**

The number $\frac{54}{20} = 2.7$.

b. **Use long division to determine the decimal expansion of $\frac{7}{8}$.**

The number $\frac{7}{8} = 0.875$.

c. **Use long division to determine the decimal expansion of $\frac{8}{9}$.**

The number $\frac{8}{9} = 0.8888\ldots$.

d. **Use long division to determine the decimal expansion of $\frac{22}{7}$.**

The number $\frac{22}{7} = 3.142857\ldots$.

MP.2

e. **What do you notice about the decimal expansions of parts (a) and (b) compared to the decimal expansions of parts (c) and (d)?**

The decimal expansions of parts (a) and (b) ended. That is, when I did the long division, I was able to stop after a few steps. That was different from the work I had to do in parts (c) and (d). In part (c), I noticed that the same number kept coming up in the steps of the division, but it kept going on. In part (d), when I did the long division, it did not end. I stopped dividing after I found a few decimal digits of the decimal expansion.

Discussion (5 minutes)

Use the discussion below to elicit a dialogue about finite and infinite decimals that may not have come up in the debrief of the Opening Exercise and to prepare students for what is covered in this lesson in particular (i.e., writing fractions as finite decimals without using long division).

- Every number has a decimal expansion. That is, every number is equal to a decimal. For example, the numbers $\sqrt{3}$ and $\frac{17}{125}$ have decimal expansions. The decimal expansion of $\sqrt{3}$ will be covered in a later lesson. For now, we will focus on the decimal expansion of a number like $\frac{17}{125}$ and whether it can be expressed as a finite or an infinite decimal.
- How would you classify the decimal expansions of parts (a)–(d)?
    - *Parts (a) and (b) are finite decimals, and parts (c) and (d) are infinite decimals.*
- In the context of fractions, a decimal is, by definition, a fraction with a denominator equal to a power of 10. These decimals are known as *finite decimals*. The distinction must be made because we will soon be working with infinite decimals. Can you think of any numbers that are infinite decimals?
    - *A decimal that repeats or a number like π is an infinite decimal.*

Scaffolding:

Students may benefit from a graphic organizer, shown below, that shows the key information regarding finite and infinite decimals.

Finite Decimals	Infinite Decimals
Definition: Examples:	Definition: Examples:

- Decimals that repeat, such as $0.8888888\ldots$ or $0.454545454545\ldots$, are infinite decimals and are typically abbreviated as $0.\overline{8}$ and $0.\overline{45}$, respectively. The notation indicates that the digit 8 repeats indefinitely and that the two-digit block 45 repeats indefinitely. The number π, $3.1415926535\ldots$, is also a famous infinite decimal that does not have a block of digits that repeats indefinitely.
- In Grade 7, you learned a general procedure for writing the decimal expansion of a fraction such as $\frac{5}{14}$ using long division. In the next lesson, we will closely examine the long division algorithm and why the procedure makes sense.
- Today, we will learn a method for converting a fraction to a decimal that does not require long division. Each of the fractions in the examples and exercises in *this* lesson are simplified fractions. The method we will learn requires that we begin with a simplified fraction.

MP.7

- Return to the Opening Exercise. We know that the decimals in parts (a) and (b) are finite, while the decimals in parts (c) and (d) are not. What do you notice about the denominators of these fractions that might explain this?
    - *The denominators of the fractions in parts (a) and (b) are the products of* 2*'s and* 5*'s. For example, the denominator* $20 = 2 \times 2 \times 5$*, and the denominator* $8 = 2 \times 2 \times 2$*. The denominators of the fractions in parts (c) and (d) were not the products of* 2*'s and* 5*'s. For example,* $9 = 3 \times 3$*, and* $7 = 1 \times 7$*.*

Fractions whose denominators are a product of 2's or 5's, or both, are finite decimals. Fractions like $\frac{1}{4}$, $\frac{6}{125}$, and $\frac{9}{10}$ can be expressed as finite decimals because $4 = 2^2$, $125 = 5^3$, and $10 = 2 \times 5$.

- Other fractions like $\frac{5}{14}$ cannot be expressed as a finite decimal because $14 = 2 \times 7$. Therefore, $\frac{5}{14}$ has an infinite decimal expansion.

Example 1 (4 minutes)

Example 1

Consider the fraction $\frac{5}{8}$. Is it equal to a finite decimal? How do you know?

- Consider the fraction $\frac{5}{8}$. Is it equal to a finite decimal? How do you know?
    - *The fraction* $\frac{5}{8}$ *is equal to a finite decimal because the denominator* 8 *is a product of* 2*'s. Specifically,* $8 = 2^3$.
- Since we know that the fraction $\frac{5}{8}$ is equal to a finite decimal, then we can find a fraction $\frac{k}{10^n}$, where k and n are positive integers, that will give us the decimal value that $\frac{5}{8}$ is equal to.
- We must find positive integers k and n so that $\frac{5}{8} = \frac{k}{10^n}$.
- Explain the meaning of k and 10^n in the equation above.
    - *The number* k *will be the numerator, a positive integer, of a fraction equivalent to* $\frac{5}{8}$ *that has a denominator that is a power of* 10*, e.g.,* 10^2, 10^5*, or* 10^n.
- Recall what we learned about the laws of exponents in Module 1: $(ab)^n = a^n b^n$. We will now put that knowledge to use.

  We know that $8 = 2^3$ and $10^n = (2 \times 5)^n = 2^n \times 5^n$. Comparing the denominators of the fractions, $2^3 \times 5^n = 2^n \times 5^n = 10^n$.

  What must n equal?
    - n *must be* 3.
- To rewrite the fraction $\frac{5}{8}$ so that it has a denominator of the form 10^n, we must multiply 2^3 by 5^3. Based on what you know about equivalent fractions, by what must we multiply the numerator of $\frac{5}{8}$?
    - *To make an equivalent fraction, we will need to multiply the numerator by* 5^3 *also.*

- By equivalent fractions:

$$\frac{5}{8} = \frac{5}{2^3} = \frac{5 \times 5^3}{2^3 \times 5^3} = \frac{5^4}{(2 \times 5)^3} = \frac{625}{10^3},$$

  where $k = 625$ and $n = 3$, and both are positive integers.

- Using the fraction $\frac{625}{10^3}$, we can write the decimal value of $\frac{5}{8}$. What is it? Explain.
  - $\frac{5}{8} = 0.625$ *because* $\frac{625}{10^3} = \frac{625}{1000}$. *Using what we know about place value, we have six hundred twenty-five thousandths, or* 0.625.

Example 2 (4 minutes)

Example 2

Consider the fraction $\frac{17}{125}$. Is it equal to a finite or an infinite decimal? How do you know?

- Let's consider the fraction $\frac{17}{125}$ mentioned earlier. We want the decimal value of this number. Is it a finite or an infinite decimal? How do you know?
  - *We know that the fraction* $\frac{17}{125}$ *is equal to a finite decimal because the denominator* 125 *is a product of* 5*'s. Specifically,* $125 = 5^3$.
- What will we need to multiply 5^3 by so that it is equal to $(2 \times 5)^n = 10^n$?
  - *We will need to multiply by* 2^3 *so that* $2^3 \times 5^3 = (2 \times 5)^3 = 10^3$.
- Begin with $\frac{17}{125} = \frac{17}{5^3}$. Use what you know about equivalent fractions to rewrite $\frac{17}{125} = \frac{k}{10^n}$, and then write the decimal form of the fraction.
  - $\frac{17}{125} = \frac{17}{5^3} = \frac{17 \times 2^3}{2^3 \times 5^3} = \frac{17 \times 8}{(2 \times 5)^3} = \frac{136}{10^3} = 0.136$

Exercises 1–5 (5 minutes)

Students complete Exercises 1–5 independently.

Exercises 1–5

Show your steps, but use a calculator for the multiplication.

1. **Convert the fraction $\frac{7}{8}$ to a decimal.**
   a. **Write the denominator as a product of 2's or 5's. Explain why this way of rewriting the denominator helps to find the decimal representation of $\frac{7}{8}$.**

      The denominator $8 = 2^3$. It is helpful to know that $8 = 2^3$ because it shows how many factors of 5 will be needed to multiply the numerator and denominator by so that an equivalent fraction is produced with a denominator that is a multiple of 10. When the denominator is a multiple of 10, the fraction can easily be written as a decimal using what I know about place value.

b. Find the decimal representation of $\frac{7}{8}$. Explain why your answer is reasonable.

$$\frac{7}{8}=\frac{7}{2^3}=\frac{7\times 5^3}{2^3\times 5^3}=\frac{875}{10^3}=0.875$$

The answer is reasonable because the decimal value, 0.875*, is less than* 1 *just like the fraction* $\frac{7}{8}$*. Also, it is reasonable and correct because the fraction* $\frac{875}{1000}=\frac{7}{8}$*; therefore, it has the decimal expansion* 0.875*.*

2. Convert the fraction $\frac{43}{64}$ to a decimal.

The denominator $64=2^6$*.*

$$\frac{43}{64}=\frac{43}{2^6}=\frac{43\times 5^6}{2^6\times 5^6}=\frac{671\,875}{10^6}=0.671875$$

3. Convert the fraction $\frac{29}{125}$ to a decimal.

The denominator $125=5^3$*.*

$$\frac{29}{125}=\frac{29}{5^3}=\frac{29\times 2^3}{5^3\times 2^3}=\frac{232}{10^3}=0.232$$

4. Convert the fraction $\frac{19}{34}$ to a decimal.

Using long division, $\frac{19}{34}=0.5588235\ldots$*.*

5. Identify the type of decimal expansion for each of the numbers in Exercises 1–4 as finite or infinite. Explain why their decimal expansion is such.

We know that the number $\frac{7}{8}$ *had a finite decimal expansion because the denominator* 8 *is a product of* 2*'s. We know that the number* $\frac{43}{64}$ *had a finite decimal expansion because the denominator* 64 *is a product of* 2*'s. We know that the number* $\frac{29}{125}$ *had a finite decimal expansion because the denominator* 125 *is a product of* 5*'s. We know that the number* $\frac{19}{34}$ *had an infinite decimal expansion because the denominator was not a product of* 2*'s or* 5*'s; it had a factor of* 17*.*

Example 3 (4 minutes)

Example 3

Write $\frac{7}{80}$ as a decimal. Will it be finite or infinite? Explain.

- Let's write $\frac{7}{80}$ as a decimal. Will it be finite or infinite? Explain.
    - *We know that the fraction $\frac{7}{80}$ is equal to a finite decimal because the denominator 80 is a product of 2's and 5's. Specifically, $80 = 2^4 \times 5$.*
- What will we need to multiply $2^4 \times 5$ by so that it is equal to $(2 \times 5)^n = 10^n$?
    - *We will need to multiply by 5^3 so that $2^4 \times 5^4 = (2 \times 5)^4 = 10^4$.*
- Begin with $\frac{7}{80} = \frac{7}{2^4 \times 5}$. Use what you know about equivalent fractions to rewrite $\frac{7}{80} = \frac{k}{10^n}$, and then write the decimal form of the fraction.
    - $\frac{7}{80} = \frac{7}{2^4 \times 5} = \frac{7 \times 5^3}{2^4 \times 5 \times 5^3} = \frac{7 \times 125}{(2 \times 5)^4} = \frac{875}{10^4} = 0.0875$

Example 4 (4 minutes)

Example 4

Write $\frac{3}{160}$ as a decimal. Will it be finite or infinite? Explain.

- Let's write $\frac{3}{160}$ as a decimal. Will it be finite or infinite? Explain.
    - *We know that the fraction $\frac{3}{160}$ is equal to a finite decimal because the denominator 160 is a product of 2's and 5's. Specifically, $160 = 2^5 \times 5$.*
- What will we need to multiply $2^5 \times 5$ by so that it is equal to $(2 \times 5)^n = 10^n$?
    - *We will need to multiply by 5^4 so that $2^5 \times 5^5 = (2 \times 5)^5 = 10^5$.*
- Begin with $\frac{3}{160} = \frac{3}{2^5 \times 5}$. Use what you know about equivalent fractions to rewrite $\frac{3}{160} = \frac{k}{10^n}$, and then write the decimal form of the fraction.
    - $\frac{3}{160} = \frac{3}{2^5 \times 5} = \frac{3 \times 5^4}{2^5 \times 5 \times 5^4} = \frac{3 \times 625}{(2 \times 5)^5} = \frac{1875}{10^5} = 0.01875$

Exercises 6–8 (5 minutes)

Students complete Exercises 6–8 independently.

Exercises 6–8

Show your steps, but use a calculator for the multiplication.

6. **Convert the fraction $\frac{37}{40}$ to a decimal.**
    a. **Write the denominator as a product of 2's and/or 5's. Explain why this way of rewriting the denominator helps to find the decimal representation of $\frac{37}{40}$.**

    The denominator $40 = 2^3 \times 5$. It is helpful to know that $40 = 2^3 \times 5$ because it shows by how many factors of 5 the numerator and denominator will need to be multiplied to produce an equivalent fraction with a denominator that is a multiple of 10. When the denominator is a multiple of 10, the fraction can easily be written as a decimal using what I know about place value.

b. **Find the decimal representation of $\frac{37}{40}$. Explain why your answer is reasonable.**

$$\frac{37}{40}=\frac{37}{2^3\times 5}=\frac{37\times 5^2}{2^3\times 5\times 5^2}=\frac{925}{10^3}=0.925$$

The answer is reasonable because the decimal value, 0.925, is less than 1 just like the fraction $\frac{37}{40}$. Also, it is reasonable and correct because the fraction $\frac{925}{1000}=\frac{37}{40}$; therefore, it has the decimal expansion 0.925.

7. **Convert the fraction $\frac{3}{250}$ to a decimal.**

The denominator $250=2\times 5^3$.

$$\frac{3}{250}=\frac{3}{2\times 5^3}=\frac{3\times 2^2}{2\times 2^2\times 5^3}=\frac{12}{10^3}=0.012$$

8. **Convert the fraction $\frac{7}{1250}$ to a decimal.**

The denominator $1250=2\times 5^4$.

$$\frac{7}{1250}=\frac{7}{2\times 5^4}=\frac{7\times 2^3}{2\times 2^3\times 5^4}=\frac{56}{10^4}=0.0056$$

Closing (3 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that finite decimals are fractions with denominators that can be expressed as products of 2's and 5's.
- We know how to use equivalent fractions to convert a fraction to its decimal equivalent.
- We know that infinite decimals are those that repeat, like $0.\overline{3}$, or those that neither repeat nor terminate, such as π.

Lesson Summary

Fractions with denominators that can be expressed as products of 2's and/or 5's have decimal expansions that are finite.

Example:

Does the fraction $\frac{1}{8}$ have a finite or an infinite decimal expansion?

Since $8 = 2^3$, then the fraction has a finite decimal expansion. The decimal expansion is found by

$$\frac{1}{8} = \frac{1}{2^3} = \frac{1 \times 5^3}{2^3 \times 5^3} = \frac{125}{10^3} = 0.125.$$

When the denominator of a fraction cannot be expressed as a product of 2's and/or 5's, then the decimal expansion of the number will be infinite.

When infinite decimals repeat, such as $0.8888888 \ldots$ or $0.454545454545 \ldots$, they are typically abbreviated using the notation $0.\overline{8}$ and $0.\overline{45}$, respectively. The notation indicates that the digit 8 repeats indefinitely and that the two-digit block 45 repeats indefinitely.

Exit Ticket (4 minutes)

Name ______________________________ Date ____________

Lesson 6: Finite and Infinite Decimals

Exit Ticket

Convert each fraction to a finite decimal. If the fraction cannot be written as a finite decimal, then state how you know. Show your steps, but use a calculator for the multiplication.

1. $\frac{9}{16}$

2. $\frac{8}{125}$

3. $\frac{4}{15}$

4. $\frac{1}{200}$

Exit Ticket Sample Solutions

Convert each fraction to a finite decimal. If the fraction cannot be written as a finite decimal, then state how you know. Show your steps, but use a calculator for the multiplication.

1. $\frac{9}{16}$

   The denominator $16 = 2^4$.

$$\frac{9}{16} = \frac{9}{2^4} = \frac{9 \times 5^4}{2^4 \times 5^4} = \frac{9 \times 625}{10^4} = \frac{5625}{10^4} = 0.5625$$

2. $\frac{8}{125}$

   The denominator $125 = 5^3$.

$$\frac{8}{125} = \frac{8}{5^3} = \frac{8 \times 2^3}{5^3 \times 2^3} = \frac{8 \times 8}{10^3} = \frac{64}{10^3} = 0.064$$

3. $\frac{4}{15}$

   The fraction $\frac{4}{15}$ *is not a finite decimal because the denominator* $15 = 5 \times 3$. *Since the denominator cannot be expressed as a product of* 2*'s and* 5*'s, then* $\frac{4}{15}$ *is not a finite decimal.*

4. $\frac{1}{200}$

   The denominator $200 = 2^3 \times 5^2$.

$$\frac{1}{200} = \frac{1}{2^3 \times 5^2} = \frac{1 \times 5}{2^3 \times 5^2 \times 5} = \frac{5}{2^3 \times 5^3} = \frac{5}{10^3} = 0.005$$

Problem Set Sample Solutions

Convert each fraction to a finite decimal. If the fraction cannot be written as a finite decimal, then state how you know. Show your steps, but use a calculator for the multiplication.

1. $\frac{2}{32}$

   The fraction $\frac{2}{32}$ *simplifies to* $\frac{1}{16}$.

   The denominator $16 = 2^4$.

$$\frac{1}{16} = \frac{1}{2^4} = \frac{1 \times 5^4}{2^4 \times 5^4} = \frac{625}{10^4} = 0.0625$$

2. $\frac{99}{125}$

   a. Write the denominator as a product of 2's and/or 5's. Explain why this way of rewriting the denominator helps to find the decimal representation of $\frac{99}{125}$.

   The denominator $125 = 5^3$. *It is helpful to know that* $125 = 5^3$ *because it shows by how many factors of* 2 *the numerator and denominator will need to be multiplied to produce an equivalent fraction with a denominator that is a multiple of* 10. *When the denominator is a multiple of* 10, *the fraction can easily be written as a decimal using what I know about place value.*

   b. Find the decimal representation of $\frac{99}{125}$. Explain why your answer is reasonable.

$$\frac{99}{125} = \frac{99}{5^3} = \frac{99 \times 2^3}{2^3 \times 5^3} = \frac{792}{10^3} = 0.792$$

   The answer is reasonable because the decimal value, 0.792, *is less than* 1 *just like the fraction* $\frac{99}{125}$. *Also, it is reasonable and correct because the fraction* $\frac{792}{1000} = \frac{99}{125}$; *therefore, it has the decimal expansion* 0.792.

3. $\frac{15}{128}$

   The denominator $128 = 2^7$.

$$\frac{15}{128} = \frac{15}{2^7} = \frac{15 \times 5^7}{2^7 \times 5^7} = \frac{1171875}{10^7} = 0.1171875$$

4. $\frac{8}{15}$

   The fraction $\frac{8}{15}$ *is not a finite decimal because the denominator* $15 = 3 \times 5$. *Since the denominator cannot be expressed as a product of* 2*'s and* 5*'s, then* $\frac{8}{15}$ *is not a finite decimal.*

5. $\frac{3}{28}$

   The fraction $\frac{3}{28}$ *is not a finite decimal because the denominator* $28 = 2^2 \times 7$. *Since the denominator cannot be expressed as a product of* 2*'s and* 5*'s, then* $\frac{3}{28}$ *is not a finite decimal.*

6. $\frac{13}{400}$

   The denominator $400 = 2^4 \times 5^2$.

$$\frac{13}{400} = \frac{13}{2^4 \times 5^2} = \frac{13 \times 5^2}{2^4 \times 5^2 \times 5^2} = \frac{325}{10^4} = 0.0325$$

7. $\dfrac{5}{64}$

The denominator $64 = 2^6$.

$$\frac{5}{64} = \frac{5}{2^6} = \frac{5 \times 5^6}{2^6 \times 5^6} = \frac{78125}{10^6} = 0.078125$$

8. $\dfrac{15}{35}$

The fraction $\frac{15}{35}$ *reduces to* $\frac{3}{7}$. *The denominator* 7 *cannot be expressed as a product of* 2*'s and* 5*'s. Therefore,* $\frac{3}{7}$ *is not a finite decimal.*

9. $\dfrac{199}{250}$

The denominator $250 = 2 \times 5^3$.

$$\frac{199}{250} = \frac{199}{2 \times 5^3} = \frac{199 \times 2^2}{2 \times 2^2 \times 5^3} = \frac{796}{10^3} = 0.796$$

10. $\dfrac{219}{625}$

The denominator $625 = 5^4$.

$$\frac{219}{625} = \frac{219}{5^4} = \frac{219 \times 2^4}{2^4 \times 5^4} = \frac{3504}{10^4} = 0.3504$$

Lesson 7: Infinite Decimals

Student Outcomes

- Students know the intuitive meaning of an infinite decimal.
- Students will be able to explain why the infinite decimal $0.\overline{9}$ is equal to 1.

Lesson Notes

The purpose of this lesson is to show the connection between the various forms of a number, specifically the decimal expansion, the expanded form of a decimal, and a visual representation on the number line. Given the decimal expansion of a number, students use what they know about place value to write the expanded form of the number. That expanded form is then shown on the number line by looking at increasingly smaller intervals of 10, beginning with tenths, then hundredths, then thousandths, and so on. The strategy of examining increasingly smaller intervals of negative powers of 10 is how students will learn to write the decimal expansions of irrational numbers.

Classwork

Opening Exercise (7 minutes)

MP.3

Opening Exercise

a. **Write the expanded form of the decimal 0.3765 using powers of 10.**

$$0.3765 = \frac{3}{10} + \frac{7}{10^2} + \frac{6}{10^3} + \frac{5}{10^4}$$

b. **Write the expanded form of the decimal $0.3333333\ldots$ using powers of 10.**

$$0.333333\ldots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \frac{3}{10^6} + \cdots$$

c. **What is an infinite decimal? Give an example.**

An infinite decimal is a decimal with digits that do not end. They may repeat, but they never end. An example of an infinite decimal is $0.333333\ldots$.

d. **Do you think it is acceptable to write that $1 = 0.99999\ldots$? Why or why not?**

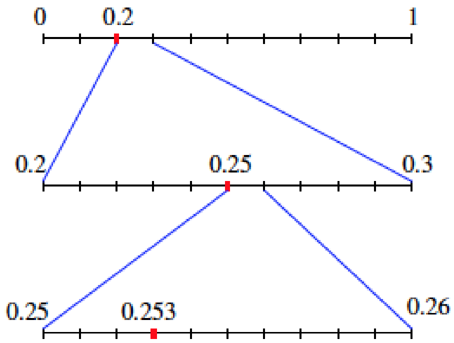
Answers will vary. Have a brief discussion with students about this exercise. The answer will be revisited in the discussion below.

Discussion (20 minutes)

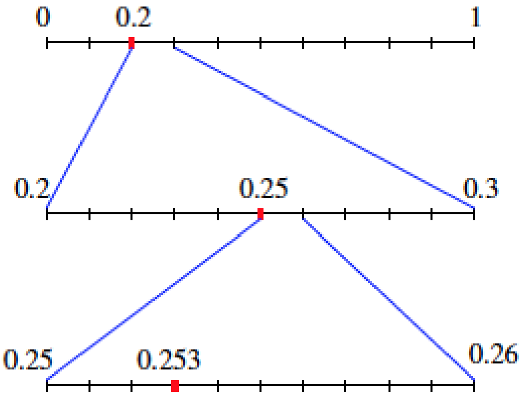
Example 1

Example 1

The number 0.253 is represented on the number line below.



- Each decimal digit is another division of a power of 10. Visually, the number 0.253 can be represented first as the segment from 0 to 1, divided into 10 equal parts, noting the first division as 0.2. Then, the segment from 0.2 to 0.3 is divided into 10 equal parts, noting the fifth division as 0.25. Then, the segment from 0.25 to 0.26 is divided into 10 equal parts, noting the third division as 0.253.



- What we have done here is represented increasingly smaller increments of negative powers of 10: $\frac{2}{10}$, then $\frac{25}{10^2}$, and finally $\frac{253}{10^3}$.
- Now, consider the expanded form of the decimal with denominators that are powers of 10; that is, $\frac{1}{10^n}$ where n is a whole number. The finite decimal can be represented in three steps:

  The first decimal digit, $0.2 = \frac{2}{10}$.

  The first two decimal digits, $0.25 = \frac{2}{10} + \frac{5}{10^2} = \frac{25}{10^2}$.

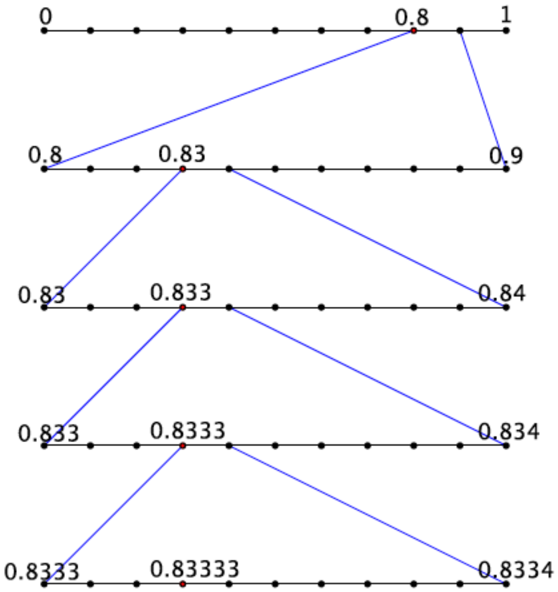
  The first three decimal digits, $0.253 = \frac{2}{10} + \frac{5}{10^2} + \frac{3}{10^3} = \frac{253}{10^3}$.

- The number 0.253 can be completely represented because there are a finite number of decimal digits. The value of the number 0.253 can clearly be represented by the fraction $\frac{253}{10^3}$; that is, $\frac{253}{1000} = 0.253$.
- Explain how 0.253, the number lines, and the expanded form of the number are related.
    - *The number 0.253 is equal to the sum of the following fractions: $\frac{2}{10} + \frac{5}{10^2} + \frac{3}{10^3}$. Then, $0.253 = \frac{2}{10} + \frac{5}{10^2} + \frac{3}{10^3}$. The first number line above shows the first term of the sum, $\frac{2}{10}$. When the interval from 0.2 to 0.3 is examined in hundredths, we can locate the second term of the sum, $\frac{5}{10^2}$, and specifically the sum of the first two terms $\frac{2}{10} + \frac{5}{10^2} = \frac{25}{100}$. Then, the interval between 0.25 and 0.26 is examined in thousandths. We can then locate the third term of the sum, $\frac{3}{10^3}$, and specifically the entire sum of the expanded form of 0.253, which is $\frac{253}{1000}$.*
- What do you think the sequence would look like for an infinite decimal?
    - *The sequence for an infinite decimal would never end; it would go on infinitely.*

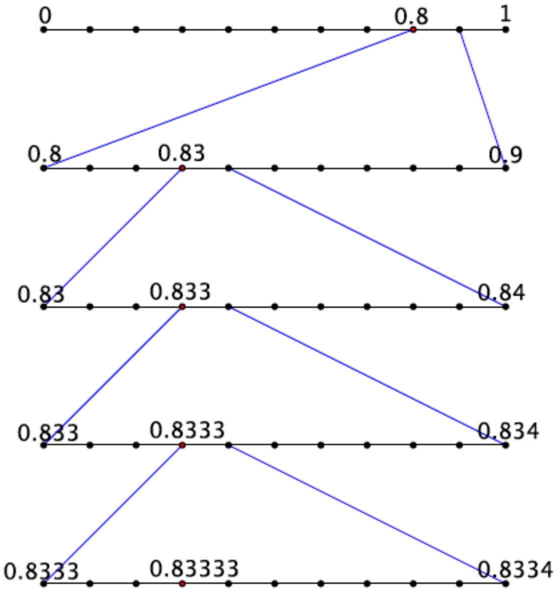
Example 2

Example 2

The number $\frac{5}{6} = 0.833333\ldots = 0.8\overline{3}$ is represented on the number line below.



- Now, consider the equality $\frac{5}{6} = 0.833333\ldots = 0.8\overline{3}$. Notice that at the second step, the work begins to repeat, which coincides with the fact that the decimal digit of 3 repeats.



- What is the expanded form of the decimal $0.833333\ldots$?
  - $0.833333\ldots = \frac{8}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \frac{3}{10^6} + \cdots$
- We see again that at the second step the work begins to repeat.
- Each step can be represented by increasing powers of 10 in the denominator: $\frac{8}{10}, \frac{83}{10^2}, \frac{833}{10^3}, \frac{8333}{10^4}, \frac{83333}{10^5}, \frac{833333}{10^6}$, and so on. When will it end? Explain.
  - *It will never end because the decimal is infinite.*
- Notice that in the last few steps, the value of the number being represented gets increasingly smaller. For example, in the sixth step, we have included $\frac{3}{10^6}$ more of the value of the number. That is 0.000003. As the steps increase, we are dealing with incrementally smaller numbers that approach a value of 0. Consider the 20th step; we would be adding $\frac{3}{10^{20}}$ to the value of the number, which is 0.00000000000000000003. It should be clear that $\frac{3}{10^{20}}$ is a very small number and is fairly close to a value of 0.

Scaffolding:

The words *finite* and *infinite* would appear to have similar pronunciations, when in fact the stresses and vowel sounds are different. Helping students make the text or speech connection will be useful so that they recognize the words when written and when spoken.

- At this point in our learning, we know how to convert a fraction to a decimal, even if it is infinite. How do we do that?
  - *We use long division when the fraction is equal to an infinite decimal.*
- We will soon learn how to write an infinite decimal as a fraction; in other words, we will learn how to convert a number in the form of $0.8\overline{3}$ to a fraction, $\frac{5}{6}$.
- Now, back to Opening Exercise, part (d). Is it acceptable to write that $1 = 0.9999999\ldots$? With an increased understanding of infinite decimals, have you changed your mind about whether or not this is an acceptable statement?

Have a discussion with students about Opening Exercise, part (d). If students have changed their minds, ask them to explain why.

- When you consider the infinite steps that represent the decimal $0.9999999\ldots$, it is clear that the value we add with each step is an increasingly smaller value, so it makes sense to write that $0.\overline{9} = 1$.
- A concern may be that the left side is not really equal to1; it only gets closer and closer to 1. However, such a statement confuses the process of representing a finite decimal with an infinite decimal. That is, as we increase the steps, we are adding smaller and smaller values to the number. It is so small that the amount we add is practically zero. That means with each step, we are showing that the number $0.\overline{9}$ is getting closer and closer to 1. Since the process is infinite, it is acceptable to write $0.\overline{9} = 1$.

MP.3

Provide students time to convince a partner that $0.\overline{9} = 1$. Encourage students to be as critical as possible. Select a student to share her argument with the class.

- In many (but not all) situations, we often treat infinite decimals as finite decimals. We do this for the sake of computation. Imagine multiplying the infinite decimal $0.83333333\ldots$ by any other number or even another infinite decimal. To do this work precisely, you would never finish writing one of the infinite decimals, let alone perform the multiplication. For this reason, we often shorten the infinite decimal using the repeat block as our guide for performing operations.

MP.3

- Every finite decimal is the sum of a whole number (which could be zero) and a finite decimal that is less than 1. Show that this is true for the number 3.141592.
    - *The number* 3.141592 *is equal to the whole number* 3 *plus the finite decimal* 0.141592: $3.141592 = 3 + 0.141592$.
- By definition of a finite decimal (one whose denominator can be expressed as a product of 2's and 5's), the number 3.141592 is equivalent to

$$\begin{aligned}\frac{3141592}{10^6} &= \frac{(3\times 10^6)+141592}{10^6}\\ &= \frac{3\times 10^6}{10^6}+\frac{141592}{10^6}\\ &= 3+\frac{141592}{10^6}\\ &= 3+0.141592.\end{aligned}$$

- We will soon claim that every infinite decimal is the sum of a whole number and an infinite decimal that is less than 1. Consider the infinite decimal $3.141592\ldots$.

$$3.141592\ldots = 3 + 0.141592\ldots$$

This fact will help us to write an infinite decimal as a fraction in Lesson 10.

Exercises 1–6 (8 minutes)

Students complete Exercises 1–6 independently or in pairs.

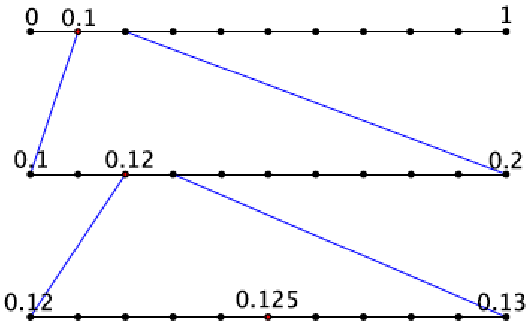
Exercises 1–6

1.

a. Write the expanded form of the decimal 0.125 using powers of 10.

$$\mathbf{0.125 = \frac{1}{10} + \frac{2}{10^2} + \frac{5}{10^3}}$$

b. **Show on the number line the representation of the decimal 0.125.**



c. **Is the decimal finite or infinite? How do you know?**

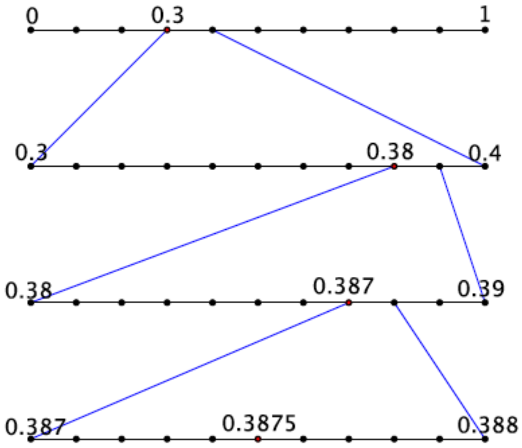
The decimal 0.125 *is finite because it can be completely represented by a finite number of steps.*

2.

a. **Write the expanded form of the decimal 0.3875 using powers of 10.**

$$0.3875 = \frac{3}{10} + \frac{8}{10^2} + \frac{7}{10^3} + \frac{5}{10^4}$$

b. **Show on the number line the representation of the decimal 0.3875.**



c. **Is the decimal finite or infinite? How do you know?**

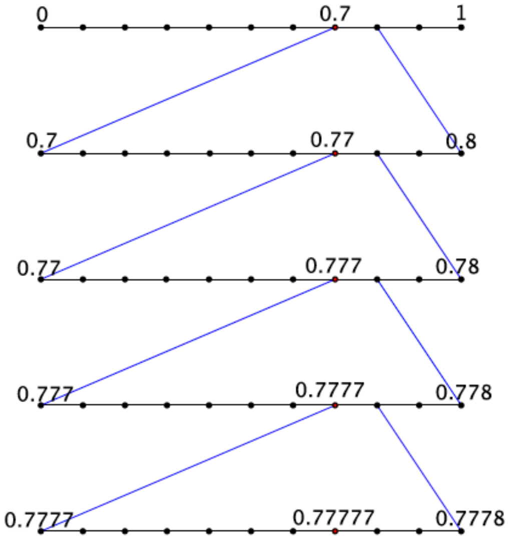
The decimal 0.3875 *is finite because it can be completely represented by a finite number of steps.*

3.

a. **Write the expanded form of the decimal $0.777777\ldots$ using powers of 10.**

$$0.777777\ldots = \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \frac{7}{10^4} + \frac{7}{10^5} + \frac{7}{10^6} + \cdots$$

b. Show on the number line the representation of the decimal $0.777777\ldots$.



c. Is the decimal finite or infinite? How do you know?

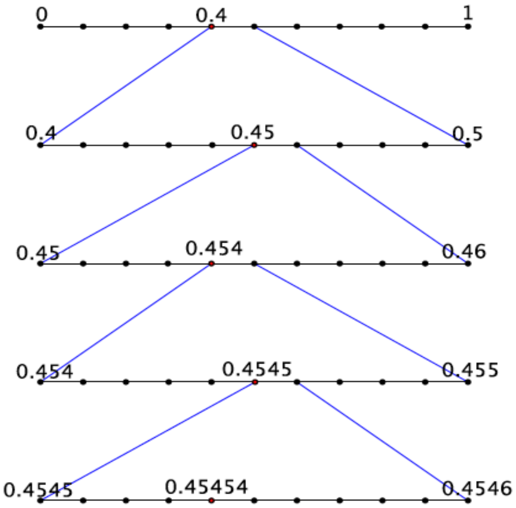
The decimal $0.777777\ldots$ is infinite because it cannot be represented by a finite number of steps. Because the number 7 continues to repeat, there will be an infinite number of steps in the sequence.

4\.

a. Write the expanded form of the decimal $0.\overline{45}$ using powers of 10.

$$0.\overline{45} = \frac{4}{10} + \frac{5}{10^2} + \frac{4}{10^3} + \frac{5}{10^4} + \frac{4}{10^5} + \frac{5}{10^6} + \cdots$$

b. Show on the number line the representation of the decimal $0.\overline{45}$.



c. **Is the decimal finite or infinite? How do you know?**

The decimal $0.\overline{45}$ *is infinite because it cannot be represented by a finite number of steps. Because the digits* 4 *and* 5 *continue to repeat, there will be an infinite number of steps in the sequence.*

5. **Order the following numbers from least to greatest:** 2.121212, 2.1, 2.2, **and** $2.\overline{12}$.

2.1, 2.121212, $2.\overline{12}$, *and* 2.2

6. **Explain how you knew which order to put the numbers in.**

Each number is the sum of the whole number 2 *and a decimal. When you write each number in this manner, you get the following expansions.*

$$2.121212 = 2 + \frac{1}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} + \frac{2}{10^6}$$

$$2.1 = 2 + \frac{1}{10}$$

$$2.2 = 2 + \frac{2}{10}$$

$$2.\overline{12} = 2 + \frac{1}{10} + \frac{2}{10^2} + \frac{1}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} + \frac{2}{10^6} + \frac{1}{10^7} + \frac{2}{10^8} + \cdots$$

In this form, it is clear that 2.1 *is the least of the four numbers, followed by the finite decimal* 2.121212*, then the infinite decimal* $2.\overline{12}$*, and finally* 2.2*.*

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that an infinite decimal is a decimal whose expanded form and number line representation is infinite.
- We know that each step in the sequence of an infinite decimal adds an increasingly smaller value to the number, so small that the amount approaches zero.
- We know that the infinite decimal $0.\overline{9} = 1$ and can explain why this is true.

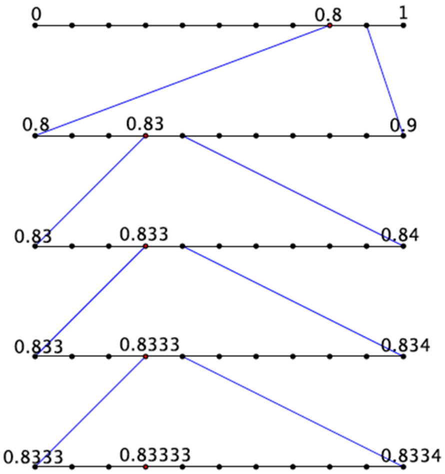
Lesson Summary

An infinite decimal is a decimal whose expanded form and number line representation are infinite.

Example:

The expanded form of the decimal $0.83333\ldots$ is $0.8\overline{3} = \frac{8}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \cdots$.

The number is represented on the number line shown below. Each new line is a magnification of the interval shown above it. For example, the first line is the unit from 0 to 1 divided into ten equal parts, or tenths. The second line is the interval from 0.8 to 0.9 divided into ten equal parts, or hundredths. The third line is the interval from 0.83 to 0.84 divided into ten equal parts, or thousandths, and so on.



With each new line, we are representing an increasingly smaller value of the number, so small that the amount approaches a value of zero. Consider the 20th line of the picture above. We would be adding $\frac{3}{10^{20}}$ to the value of the number, which is 0.00000000000000000003. It should be clear that $\frac{3}{10^{20}}$ is a very small number and is fairly close to a value of zero.

This reasoning is what we use to explain why the value of the infinite decimal $0.\overline{9}$ is 1.

Exit Ticket (5 minutes)

There are three items as part of the Exit Ticket, but it may be necessary to only use the first two to assess students' understanding.

Name ______________________________ Date ______________

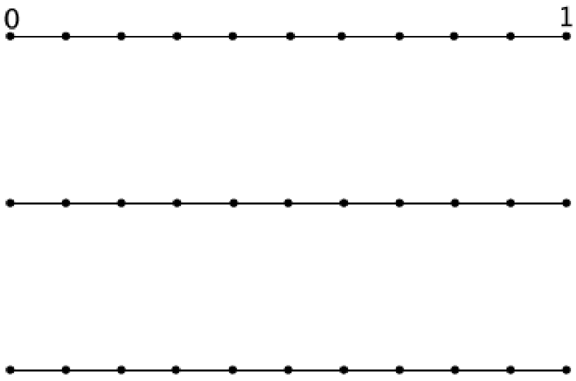
Lesson 7: Infinite Decimals

Exit Ticket

1.

a. Write the expanded form of the decimal 0.829 using powers of 10.

b. Show on the number line the representation of the decimal 0.829.

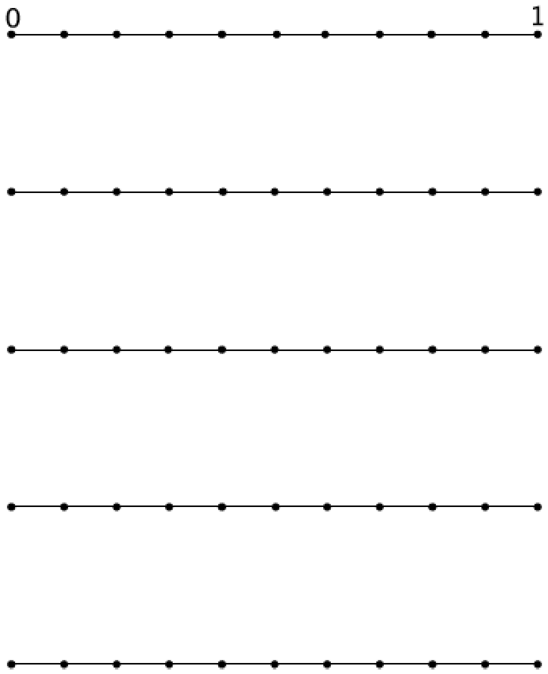


c. Is the decimal finite or infinite? How do you know?

2.

a. Write the expanded form of the decimal $0.55555\ldots$ using powers of 10.

b. Show on the number line the representation of the decimal $0.555555\ldots$.

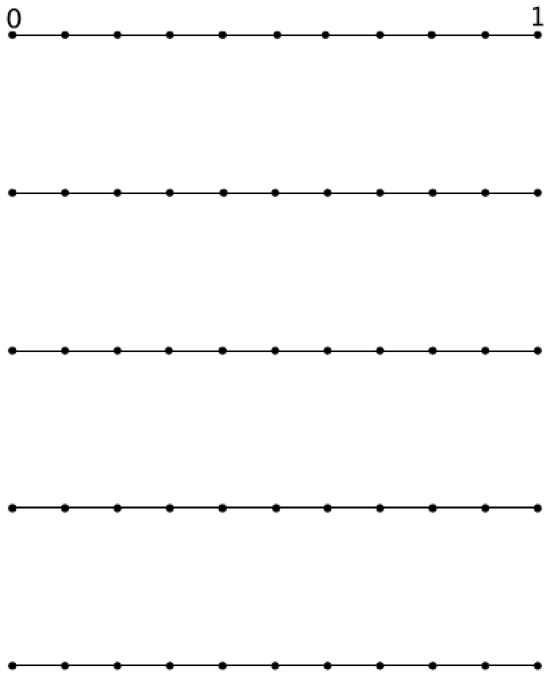


c. Is the decimal finite or infinite? How do you know?

3.

a. Write the expanded form of the decimal $0.\overline{573}$ using powers of 10.

b. Show on the number line the representation of the decimal $0.\overline{573}$.



c. Is the decimal finite or infinite? How do you know?

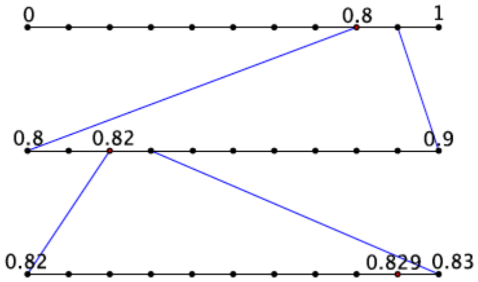
Exit Ticket Sample Solutions

1.

a. **Write the expanded form of the decimal 0.829 using powers of 10.**

$$0.829 = \frac{8}{10} + \frac{2}{10^2} + \frac{9}{10^3}$$

b. **Show on the number line the representation of the decimal 0.829.**



c. **Is the decimal finite or infinite? How do you know?**

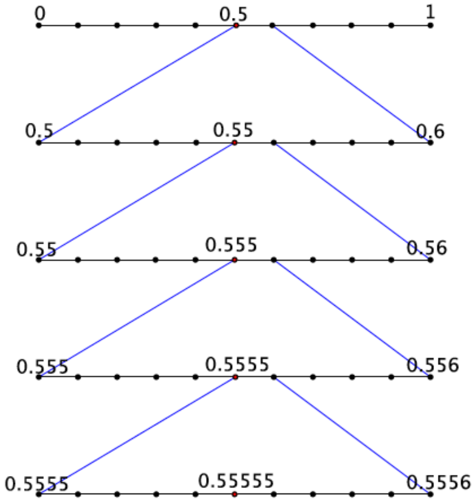
The decimal 0.829 is finite because it can be completely represented by a finite number of steps.

2.

a. **Write the expanded form of the decimal $0.55555\ldots$ using powers of 10.**

$$0.55555\ldots = \frac{5}{10} + \frac{5}{10^2} + \frac{5}{10^3} + \frac{5}{10^4} + \frac{5}{10^5} + \frac{5}{10^6} + \cdots$$

b. **Show on the number line the representation of the decimal $0.555555\ldots$.**



c. **Is the decimal finite or infinite? How do you know?**

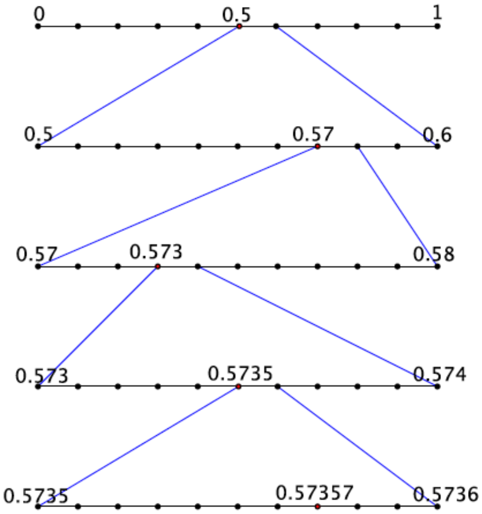
The decimal $0.555555\ldots$ *is infinite because it cannot be represented by a finite number of steps. Because the number* 5 *continues to repeat, there will be an infinite number of steps.*

3.

a. **Write the expanded form of the decimal $0.\overline{573}$ using powers of 10.**

$$0.\overline{573} = \frac{5}{10} + \frac{7}{10^2} + \frac{3}{10^3} + \frac{5}{10^4} + \frac{7}{10^5} + \frac{3}{10^6} + \cdots$$

b. **Describe the sequence that would represent the decimal $0.\overline{573}$.**



c. **Is the decimal finite or infinite? How do you know?**

The decimal $0.\overline{573}$ *is infinite because it cannot be represented by a finite number of steps. Because the digits* 5, 7, *and* 3 *continue to repeat, there will be an infinite number of steps.*

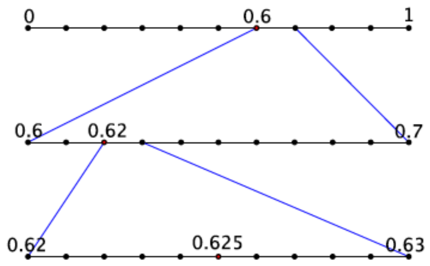
Problem Set Sample Solutions

1.

a. **Write the expanded form of the decimal 0.625 using powers of 10.**

$$0.625 = \frac{6}{10} + \frac{2}{10^2} + \frac{5}{10^3}$$

b. Show on the number line the representation of the decimal 0.625.



c. Is the decimal finite or infinite? How do you know?

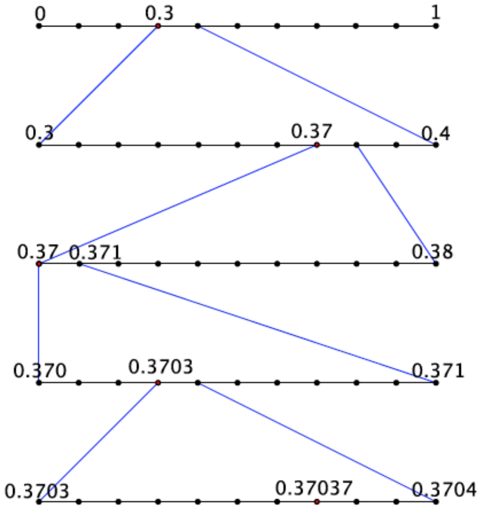
The decimal 0.625 is finite because it can be completely represented by a finite number of steps in the sequence.

2.

a. Write the expanded form of the decimal $0.\overline{370}$ using powers of 10.

$$0.\overline{370} = \frac{3}{10} + \frac{7}{10^2} + \frac{0}{10^3} + \frac{3}{10^4} + \frac{7}{10^5} + \frac{0}{10^6} + \cdots$$

b. Show on the number line the representation of the decimal $0.370370\ldots$.



c. Is the decimal finite or infinite? How do you know?

The decimal $0.\overline{370}$ is infinite because it cannot be represented by a finite number of steps. Because the digits 3, 7, and 0 continue to repeat, there will be an infinite number of steps in the sequence.

3. **Which is a more accurate representation of the number $\frac{2}{3}$: 0.6666 or $0.\overline{6}$? Explain. Which would you prefer to compute with?**

The number $\frac{2}{3}$ is more accurately represented by the decimal $0.\overline{6}$ compared to 0.6666. The long division algorithm with $\frac{2}{3}$ shows that the digit 6 repeats. Then, the expanded form of the decimal $0.\overline{6} = \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4} + \frac{6}{10^5} + \frac{6}{10^6} + \cdots$, and so on, where the number $0.6666 = \frac{6}{10} + \frac{6}{10^2} + \frac{6}{10^3} + \frac{6}{10^4}$. For this reason, $0.\overline{6}$ is more accurate. For computations, I would prefer to use 0.6666. My answer would be less precise, but at least I would be able to compute with it. When attempting to compute with an infinite number, you would never finish writing it; thus, you could never compute with it.

4. **Explain why we shorten infinite decimals to finite decimals to perform operations. Explain the effect of shortening an infinite decimal on our answers.**

We often shorten infinite decimals to finite decimals to perform operations because it would be impossible to represent an infinite decimal precisely since the sequence that describes infinite decimals has an infinite number of steps. Our answers are less precise; however, they are not that much less precise because with each additional digit in the sequence we include, we are adding a very small amount to the value of the number. The more decimals we include, the closer the value we add approaches zero. Therefore, it does not make that much of a difference with respect to our answer.

5. **A classmate missed the discussion about why $0.\overline{9} = 1$. Convince your classmate that this equality is true.**

When you consider the infinite sequence of steps that represents the decimal $0.9999999\ldots$, it is clear that the value we add with each step is an increasingly smaller value, so it makes sense to write that $0.\overline{9} = 1$. As we increase the number of steps in the sequence, we are adding smaller and smaller values to the number. Consider the 12^{th} step: 0.999999999999. The value added to the number is just 0.000000000009, which is a very small amount. The more steps that we include, the closer that value is to zero. This means that with each new step, the number $0.\overline{9}$ gets closer and closer to 1. Since this process is infinite, the number $0.\overline{9} = 1$.

6. **Explain why $0.3333 < 0.33333$.**

The number $0.3333 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4}$, and the number $0.33333 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5}$. That means that 0.33333 is exactly $\frac{3}{10^5}$ larger than 0.3333. If we examined the numbers on the number line, 0.33333 is to the right of 0.3333, meaning that it is larger than 0.3333.

Lesson 8: The Long Division Algorithm

Student Outcomes

- Students know that the long division algorithm is a basic skill to get division-with-remainder and the decimal expansion of a number in general.
- Students know why digits repeat in the decimal expansion of a fraction by using the long division algorithm.
- Students know that every rational number has a decimal expansion that repeats eventually.

Lesson Notes

In this lesson, we move toward being able to define an irrational number by formalizing the definition of a rational number.

Classwork

Example 1 (5 minutes)

Example 1

Show that the decimal expansion of $\frac{26}{4}$ is 6.5.

Scaffolding:

There is no single long division algorithm. The algorithm commonly taught and used in the U.S. is rarely used elsewhere. Students may come with earlier experiences with other division algorithms that make more sense to them. Consider using formative assessment to determine how different students approach long division.

Use the example with students so they have a model to complete Exercises 1–5.

MP.3

- Show that the decimal expansion of $\frac{26}{4}$ is 6.5.
  - *Students will most likely use the long division algorithm.*
- Division is really just another form of multiplication. Here is a demonstration of that fact: Let's consider the fraction $\frac{26}{4}$ in terms of multiplication. We want to know the greatest number of groups of 4 that are in 26. How many are there?
  - *There are 6 groups of 4 in 26.*
- Is there anything left over, a remainder?
  - *Yes, there are 2 left over.*
- Symbolically, we can express the number 26 as

$$26 = 6 \times 4 + 2.$$

MP.3

- With respect to the fraction $\frac{26}{4}$, we can represent the division as

$$\begin{aligned}\frac{26}{4} &= \frac{6\times 4+2}{4}\\ \frac{26}{4} &= \frac{6\times 4}{4}+\frac{2}{4}\\ \frac{26}{4} &= 6+\frac{2}{4}\\ \frac{26}{4} &= 6\frac{2}{4}=6\frac{1}{2}.\end{aligned}$$

- The fraction $\frac{26}{4}$ is equal to the finite decimal 6.5. When the fraction is not equal to a finite decimal, then we need to use the long division algorithm to determine the decimal expansion of the number.

Exploratory Challenge/Exercises 1–5 (15 minutes)

Students complete Exercises 1–5 independently or in pairs. The discussion that follows is related to the concepts in the exercises.

Exploratory Challenge/Exercises 1–5

1. Use long division to determine the decimal expansion of $\frac{142}{2}$.

$$\begin{array}{r} 71.0 \\ 2\overline{)142.0} \end{array}$$

a. Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{2}$.

$$\begin{aligned}142 &= \underline{\ 71\ }\times 2+\underline{\ 0\ }\\ \frac{142}{2} &= \frac{\underline{\ 71\ }\times 2+\underline{\ 0\ }}{2}\\ \frac{142}{2} &= \frac{\underline{\ 71\ }\times 2}{2}+\frac{\underline{\ 0\ }}{2}\\ \frac{142}{2} &= \underline{\ 71\ }+\frac{\underline{\ 0\ }}{2}\\ \frac{142}{2} &= \underline{\ 71.0\ }\end{aligned}$$

b. Does the number $\frac{142}{2}$ have a finite or an infinite decimal expansion? Explain how you know.

The decimal expansion of $\frac{142}{2}$ is 71.0 and is finite because the denominator of the fraction, 2, can be expressed as a product of 2's.

2. Use long division to determine the decimal expansion of $\frac{142}{4}$.

$$\begin{array}{r} 35.5 \\ 4\overline{)142.0} \end{array}$$

a. **Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{4}$.**

$$142 = \underline{\;35\;} \times 4 + \underline{\;2\;}$$

$$\frac{142}{4} = \frac{\underline{\;35\;} \times 4 + \underline{\;2\;}}{4}$$

$$\frac{142}{4} = \frac{\underline{\;35\;} \times 4}{4} + \frac{\underline{\;2\;}}{4}$$

$$\frac{142}{4} = \underline{\;35\;} + \frac{\underline{\;2\;}}{4}$$

$$\frac{142}{4} = \underline{\;35\frac{2}{4} = 35.5\;}$$

b. **Does the number $\frac{142}{4}$ have a finite or an infinite decimal expansion? Explain how you know.**

The decimal expansion of $\frac{142}{4}$ *is* 35.5 *and is finite because the denominator of the fraction,* 4*, can be expressed as a product of* 2*'s.*

3. **Use long division to determine the decimal expansion of $\frac{142}{6}$.**

$$\begin{array}{r}
23.666\\
6\,\overline{)142.000}\\
\underline{12}\\
22\\
\underline{18}\\
40\\
\underline{36}\\
40\\
\underline{36}\\
40\\
\underline{36}\\
4
\end{array}$$

a. **Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{6}$.**

$$142 = \underline{\;23\;} \times 6 + \underline{\;4\;}$$

$$\frac{142}{6} = \frac{\underline{\;23\;} \times 6 + \underline{\;4\;}}{6}$$

$$\frac{142}{6} = \frac{\underline{\;23\;} \times 6}{6} + \frac{\underline{\;4\;}}{6}$$

$$\frac{142}{6} = \underline{\;23\;} + \frac{\underline{\;4\;}}{6}$$

$$\frac{142}{6} = \underline{\;23\frac{4}{6} = 23.666\ldots\;}$$

b. **Does the number $\frac{142}{6}$ have a finite or an infinite decimal expansion? Explain how you know.**

The decimal expansion of $\frac{142}{6}$ *is* $23.666\ldots$ *and is infinite because the denominator of the fraction,* 6*, cannot be expressed as a product of* 2*'s and/or* 5*'s.*

4. Use long division to determine the decimal expansion of $\frac{142}{11}$.

$$\begin{array}{r}
12.90909 \\
11\overline{)142.00000} \\
\underline{11} \\
32 \\
\underline{22} \\
100 \\
\underline{99} \\
10 \\
\underline{00} \\
100 \\
\underline{99} \\
10 \\
\underline{00} \\
100 \\
\underline{99} \\
10
\end{array}$$

a. Fill in the blanks to show another way to determine the decimal expansion of $\frac{142}{11}$.

$$142 = \underline{\ 12\ } \times 11 + \underline{\ 10\ }$$

$$\frac{142}{11} = \frac{\underline{\ 12\ } \times 11 + \underline{\ 10\ }}{11}$$

$$\frac{142}{11} = \frac{\underline{\ 12\ } \times 11}{11} + \frac{\underline{\ 10\ }}{11}$$

$$\frac{142}{11} = \underline{\ 12\ } + \frac{\underline{\ 10\ }}{11}$$

$$\frac{142}{11} = \underline{12\frac{10}{11} = 12.90909\ldots}$$

b. Does the number $\frac{142}{11}$ have a finite or an infinite decimal expansion? Explain how you know.

The decimal expansion of $\frac{142}{11}$ *is* $12.90909\ldots$ *and is infinite because the denominator of the fraction,* 6*, cannot be expressed as a product of* 2*'s and/or* 5*'s.*

5. Which fractions produced an infinite decimal expansion? Why do you think that is?

The fractions that required the long division algorithm to determine the decimal expansion were $\frac{142}{6}$ *and* $\frac{142}{11}$*. The fact that these numbers had an infinite decimal expansion is due to the fact that the divisor was not a product of* 2*'s and/or* 5*'s compared to the first two fractions where the divisor was a product of* 2*'s and/or* 5*'s. In general, the decimal expansion of a number will be finite when the divisor, i.e., the denominator of the fraction, can be expressed as a product of* 2*'s and/or* 5*'s. Similarly, the decimal expansion will be infinite when the divisor cannot be expressed as a product of* 2*'s and/or* 5*'s.*

Discussion (10 minutes)

- What is the decimal expansion of $\frac{142}{2}$?

If students respond 71, ask them what decimal digits they could include without changing the value of the number.

  - *The fraction* $\frac{142}{2}$ *is equal to the decimal* 71.00000 *….*
- Did you need to use the long division algorithm to determine your answer? Why or why not?
  - *No, the long division algorithm was not necessary because there was a whole number of* 2*'s in* 142*.*
- What is the decimal expansion of $\frac{142}{4}$?
  - *The fraction* $\frac{142}{4}$ *is equal to the decimal* 35.5*.*
- What decimal digits could we include to the right of the 0.5 without changing the value?
  - *We could write the decimal as* 35.500000 *….*
- Did you need to use the long division algorithm to determine your answer? Why or why not?
  - *No, the long division algorithm was not necessary because* $\frac{142}{4} = 35 + \frac{2}{4}$*, and* $\frac{2}{4}$ *is a finite decimal. We could use what we learned in the last lesson to write* $\frac{2}{4}$ *as* 0.5*.*
- What is the decimal expansion of $\frac{142}{6}$?
  - *The fraction* $\frac{142}{6}$ *is equal to the decimal* 23.66666 *….*
- Did you need to use the long division algorithm to determine your answer? Why or why not?
  - *Yes, the long division algorithm was necessary because* $\frac{142}{6} = 23 + \frac{2}{3}$*, and* $\frac{2}{3}$ *is not a finite decimal.*

    Note: Some students may have recognized the fraction $\frac{2}{3}$ *as* 0.6666 *… and not used the long division algorithm to determine the decimal expansion.*
- How did you know when you could stop dividing?
  - *I knew to stop dividing because the remainder kept repeating. Specifically, when I used the long division algorithm, the number* 40 *kept appearing, and there are* 6 *groups of* 6 *in* 40*, leaving* 4 *as a remainder each time, which became* 40 *when I brought down another* 0*.*
- We represent the decimal expansion of $\frac{142}{6}$ as $23.\overline{6}$, where the line above the 6 is the *repeating block*; that is, the digit 6 repeats as we saw in the long division algorithm.
- What is the decimal expansion of $\frac{142}{11}$?
  - *The fraction* $\frac{142}{11}$ *is equal to the decimal* 12.90909090 *….*
- Did you need to use the long division algorithm to determine your answer? Why or why not?
  - *Yes, the long division algorithm was necessary because* $\frac{142}{11} = 12 + \frac{10}{11}$*, and* $\frac{10}{11}$ *is not a finite decimal.*
- How did you know when you could stop dividing?
  - *I knew to stop dividing because the remainder kept repeating. Specifically, when I used the long division algorithm, I kept getting the number* 10*, which is not divisible by* 11*, so I had to bring down another* 0*, making the number* 100*. This kept happening, so I knew to stop once I noticed the work I was doing was the same.*

- Which block of digits kept repeating?
    - *The block of digits that kept repeating was* 90.
- How do we represent the decimal expansion of $\frac{142}{11}$?
    - *The decimal expansion of* $\frac{142}{11}$ *is* $12.\overline{90}$.
- In general, we say that every rational number has a decimal expansion that repeats eventually. It is obvious by the repeat blocks that $\frac{142}{6}$ and $\frac{142}{11}$ are rational numbers. Are the numbers $\frac{142}{2}$ and $\frac{142}{4}$ rational? If so, what is their repeat block?

Provide students a minute or two to discuss in small groups what the repeat blocks for $\frac{142}{2}$ and $\frac{142}{4}$ are.

- *The decimal expansion of* $\frac{142}{2}$ *is* $71.0000\ldots$, *where the repeat block is* 0. *The decimal expansion of* $\frac{142}{4}$ *is* $35.50000\ldots$, *where the repeat block is* 0. *Since the numbers* $\frac{142}{2}$ *and* $\frac{142}{4}$ *have decimal expansions that repeat, then the numbers are rational.*

Exercises 6–10 (5 minutes)

Students complete Exercises 6–10 independently.

Exercises 6–10

6. Does the number $\frac{65}{13}$ have a finite or an infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

The number $\frac{65}{13} = \frac{5\times 13}{13} = 5$, *so it is a finite decimal. The decimal expansion of* $\frac{65}{13}$ *is* $5.0000\ldots$, *where the repeat block is* 0. *Therefore, the number is rational.*

7. Does the number $\frac{17}{11}$ have a finite or an infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

$$\frac{17}{11} = \frac{1\times 11}{11} + \frac{6}{11}$$

$$\begin{array}{rl}
 & 1.5454 \\
11 & \overline{)17.00000} \\
 & \underline{11} \\
 & 60 \\
 & \underline{55} \\
 & 50 \\
 & \underline{44} \\
 & 60 \\
 & \underline{55} \\
 & 50 \\
 & \underline{44} \\
 & 60
\end{array}$$

The number $\frac{17}{11}$ *has an infinite decimal expansion,* $1.\overline{54}$. *The block of digits* 54 *repeats. In doing the long division, I realized that the remainder of* 6 *and remainder of* 5 *kept reappearing in my work. Since the number has a repeat block, it is rational.*

8. **Does the number $\pi = 3.1415926535897$ … have a finite or an infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

   The number has an infinite decimal expansion. However, it does not have decimal digits that repeat in a block. For that reason, the number is not rational.

9. **Does the number $\frac{860}{999} = 0.860860860$ … have a finite or an infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

   The number has an infinite decimal expansion. However, the decimal expansion has a repeat block of 860. Because every rational number has a block that repeats, the number is rational.

10. **Does the number $\sqrt{2} = 1.41421356237$ … have a finite or an infinite decimal expansion? Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

    The number has an infinite decimal expansion. However, it does not have decimal digits that repeat in a block. For that reason, the number is not rational.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that the long division algorithm is a procedure that allows us to write the decimal expansion for infinite decimals.
- We know that every rational number has a decimal expansion that repeats eventually.

Lesson Summary

The long division algorithm is a procedure that can be used to determine the decimal expansion of infinite decimals.

Every rational number has a decimal expansion that repeats eventually. For example, the number 32 is rational because it has a repeat block of the digit 0 in its decimal expansion, $32.\overline{0}$. The number $\frac{1}{3}$ is rational because it has a repeat block of the digit 3 in its decimal expansion, $0.\overline{3}$. The number 0.454545 … is rational because it has a repeat block of the digits 45 in its decimal expansion, $0.\overline{45}$.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 8: The Long Division Algorithm

Exit Ticket

1. Write the decimal expansion of $\frac{125}{8}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

2. Write the decimal expansion of $\frac{13}{7}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

Exit Ticket Sample Solutions

1. **Write the decimal expansion of $\frac{125}{8}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

$$\begin{aligned}\frac{125}{8} &= \frac{15\times 8}{8}+\frac{5}{8}\\ &= 15\frac{5}{8}\end{aligned}$$

$$\begin{array}{l} \quad 15.625 \\ 8\overline{)125.000} \\ \quad \underline{0} \\ \quad 12 \\ \quad \underline{8} \\ \quad\; 45 \\ \quad\; \underline{40} \\ \quad\;\; 50 \\ \quad\;\; \underline{48} \\ \quad\;\;\; 20 \\ \quad\;\;\; \underline{16} \\ \quad\;\;\;\; 40 \\ \quad\;\;\;\; \underline{40} \\ \quad\;\;\;\;\; 0 \end{array}$$

The decimal expansion of $\frac{125}{8}$ *is* 15.625. *The number is rational because it is a finite decimal with a repeating block of* 0.

2. **Write the decimal expansion of $\frac{13}{7}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

$$\begin{aligned}\frac{13}{7} &= \frac{1\times 7}{7}+\frac{6}{7}\\ &= 1\frac{6}{7}\end{aligned}$$

$$\begin{array}{l} \quad 1.857142857142 \\ 7\overline{)13.000000000000} \\ \quad \underline{7} \\ \quad 60 \\ \quad \underline{56} \\ \quad\; 40 \\ \quad\; \underline{35} \\ \quad\;\; 50 \\ \quad\;\; \underline{49} \\ \quad\;\;\; 10 \\ \quad\;\;\; \underline{7} \\ \quad\;\;\;\; 30 \\ \quad\;\;\;\; \underline{28} \\ \quad\;\;\;\;\; 20 \\ \quad\;\;\;\;\; \underline{14} \\ \quad\;\;\;\;\;\; 60 \\ \quad\;\;\;\;\;\; \underline{56} \\ \quad\;\;\;\;\;\;\; 40 \\ \quad\;\;\;\;\;\;\; \underline{35} \\ \quad\;\;\;\;\;\;\;\; 50 \end{array}$$

The decimal expansion of $\frac{13}{7}$ *is* $1.\overline{857142}$. *The number is rational because there is a repeating block of* 857142.

Rational numbers have decimal expansions that repeat; therefore, $\frac{13}{7}$ *is a rational number.*

Problem Set Sample Solutions

1. Write the decimal expansion of $\frac{7000}{9}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

$$\begin{aligned}\frac{7000}{9} &= \frac{777 \times 9}{9} + \frac{7}{9}\\ &= 777\frac{7}{9}\end{aligned}$$

$$\begin{array}{l} \hphantom{9)}\;777.77 \\ 9\overline{)7000.00} \\ \hphantom{9)}\underline{63} \\ \hphantom{9)6}70 \\ \hphantom{9)6}\underline{63} \\ \hphantom{9)63}70 \\ \hphantom{9)63}\underline{63} \\ \hphantom{9)636}70 \\ \hphantom{9)636}\underline{63} \\ \hphantom{9)6363}70 \\ \hphantom{9)6363}\underline{63} \\ \hphantom{9)63636}70 \end{array}$$

The decimal expansion of $\frac{7000}{9}$ *is* $777.\overline{7}$. *The number is rational because it has the repeating digit of* 7. *Rational numbers have decimal expansions that repeat; therefore,* $\frac{7000}{9}$ *is a rational number.*

2. Write the decimal expansion of $\frac{6\,555\,555}{3}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.

$$\begin{aligned}\frac{6\,555\,555}{3} &= \frac{2\,185\,185 \times 3}{3} + \frac{0}{3}\\ &= 2\,185\,185\end{aligned}$$

The decimal expansion of $\frac{6\,555\,555}{3}$ *is* $2,185,185$. *The number is rational because we can write the repeating digit of* 0 *following the whole number. Rational numbers have decimal expansions that repeat; therefore,* $\frac{6\,555\,555}{3}$ *is a rational number.*

3. **Write the decimal expansion of $\frac{350\,000}{11}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

$$\begin{aligned}\frac{350\,000}{11} &= \frac{31\,818 \times 11}{11} + \frac{2}{11}\\ &= 31\,818\frac{2}{11}\end{aligned}$$

$$\begin{array}{r} 31818.18 \\ 11\overline{)350,000.00} \\ \underline{33} \\ 20 \\ \underline{11} \\ 90 \\ \underline{88} \\ 20 \\ \underline{11} \\ 90 \\ \underline{88} \\ 20 \\ \underline{11} \\ 90 \\ \underline{88} \\ 20 \end{array}$$

The decimal expansion of $\frac{350\,000}{11}$ *is* $31,818.\overline{18}$. *The number is rational because there is a repeating block of* 18. *Rational numbers have decimal expansions that repeat; therefore,* $\frac{350\,000}{11}$ *is a rational number.*

4. **Write the decimal expansion of $\frac{12\,000\,000}{37}$. Based on our definition of rational numbers having a decimal expansion that repeats eventually, is the number rational? Explain.**

$$\begin{aligned}\frac{12\,000\,000}{37} &= \frac{324\,324 \times 37}{37} + \frac{12}{37}\\ &= 324\,324\frac{12}{37}\end{aligned}$$

$$\begin{array}{r} 324324.324 \\ 37\overline{)12,000,000.000} \\ \underline{111} \\ 90 \\ \underline{74} \\ 160 \\ \underline{148} \\ 120 \\ \underline{111} \\ 90 \\ \underline{74} \\ 160 \\ \underline{148} \\ 120 \\ \underline{111} \\ 90 \\ \underline{74} \\ 160 \\ \underline{148} \\ 120 \end{array}$$

The decimal expansion of $\frac{12\,000\,000}{37}$ *is* $324,324.\overline{324}$. *The number is rational because there is a repeating block of* 324. *Rational numbers have decimal expansions that repeat; therefore,* $\frac{12\,000\,000}{37}$ *is a rational number.*

5. **Someone notices that the long division of $2,222,222$ by 6 has a quotient of $370,370$ and remainder 2 and wonders why there is a repeating block of digits in the quotient, namely 370. Explain to the person why this happens.**

$$\frac{2\,222\,222}{6} = \frac{370\,370 \times 6}{6} + \frac{2}{6}$$
$$= 370\,370\frac{2}{6}$$

$$\begin{array}{r} 370370 \\ 6\overline{)2222222} \\ \underline{18} \\ 42 \\ \underline{42} \\ 022 \\ \underline{18} \\ 42 \\ \underline{42} \\ 02 \end{array}$$

The reason that the block of digits 370 keeps repeating is because the long division algorithm leads us to perform the same division over and over again. In the algorithm shown above, we see that there are three groups of 6 in 22, leaving a remainder of 4. When we bring down the next 2, we see that there are exactly seven groups of 6 in 42. When we bring down the next 2, we see that there are zero groups of 6 in 2, leaving a remainder of 2. It is then that the process starts over because the next step is to bring down another 2, giving us 22, which is what we started with. Since the division repeats, then the digits in the quotient will repeat.

6. **Is the number $\frac{9}{11} = 0.81818181$ ... rational? Explain.**

The number appears to be rational because the decimal expansion has a repeat block of 81. Because every rational number has a block that repeats, the number is rational.

7. **Is the number $\sqrt{3} = 1.73205080$ ... rational? Explain.**

The number appears to have a decimal expansion that does not have decimal digits that repeat in a block. For that reason, this is not a rational number.

8. **Is the number $\frac{41}{333} = 0.1231231231$ ... rational? Explain.**

The number appears to be rational because the decimal expansion has a repeat block of 123. Because every rational number has a block that repeats, the number is rational.

Lesson 9: Decimal Expansions of Fractions, Part 1

Student Outcomes

- Students apply knowledge of equivalent fractions, long division, and the distributive property to write the decimal expansion of fractions.

Classwork

Opening Exercise (5 minutes)

Opening Exercise

a.

i. **We know that the fraction $\frac{5}{8}$ can be written as a finite decimal because its denominator is a product of 2's. Which power of 10 will allow us to easily write the fraction as a decimal? Explain.**

Since $8 = 2^3$, we will multiply the numerator and denominator by 5^3, which means that $2^3 \times 5^3 = 10^3$ will be the power of 10 that allows us to easily write the fraction as a decimal.

ii. **Write the equivalent fraction using the power of 10.**

$$\frac{5}{8} = \frac{5 \times 5^3}{2^3 \times 5^3} = \frac{625}{1000}$$

b.

i. **We know that the fraction $\frac{17}{125}$ can be written as a finite decimal because its denominator is a product of 5's. Which power of 10 will allow us to easily write the fraction as a decimal? Explain.**

Since $125 = 5^3$, we will multiply the numerator and denominator by 2^3, which means that $5^3 \times 2^3 = 10^3$ will be the power of 10 that allows us to easily write the fraction as a decimal.

ii. **Write the equivalent fraction using the power of 10.**

$$\frac{17}{125} = \frac{17 \times 2^3}{5^3 \times 2^3} = \frac{136}{1000}$$

Example 1 (5 minutes)

Example 1

Write the decimal expansion of the fraction $\frac{5}{8}$.

- Based on our previous work with finite decimals, we already know how to convert $\frac{5}{8}$ to a decimal. We will use this example to learn a strategy using equivalent fractions that can be applied to converting any fraction to a decimal.
- What is true about these fractions and why?

$$\frac{5}{8}, \frac{10}{16}, \text{ and } \frac{50}{80}$$

  - *The fractions are equivalent. In all cases, when the numerator and denominator of* $\frac{5}{8}$ *are multiplied by the same factor, it produces one of the other fractions. For example,* $\frac{5\times2}{8\times2}=\frac{10}{16}$ *and* $\frac{5\times10}{8\times10}=\frac{50}{80}$.
- What would happen if we chose 10^3 as this factor? We will still produce an equivalent fraction, but note how we use the factor of 10^3 in writing the decimal expansion of the fraction.

$$\begin{aligned}\frac{5}{8}&=\frac{5\times10^3}{8}\times\frac{1}{10^3}\\&=\frac{5000}{8}\times\frac{1}{10^3}\end{aligned}$$

- Now, we will use the long division algorithm and what we know about division with remainder for $\frac{5000}{8}$.

$$\begin{aligned}&=\frac{625\times8+0}{8}\times\frac{1}{10^3}\\&=\left(625+\frac{0}{8}\right)\times\frac{1}{10^3}\\&=625\times\frac{1}{10^3}\\&=\frac{625}{10^3}\\&=0.625\end{aligned}$$

- Because of our work with Opening Exercise, part (a), we knew ahead of time that using 10^3 will help us achieve our goal. However, any power of 10 would achieve the same result. Assume we used 10^5 instead. Do you think our answer would be the same?
  - *Yes, it should be the same, but I would have to do the work to check it.*
- Let's verify that our result would be the same if we used 10^5.

$$\begin{aligned}\frac{5}{8}&=\frac{5\times10^5}{8}\times\frac{1}{10^5}\\&=\frac{500\,000}{8}\times\frac{1}{10^5}\\&=\frac{62500\times8+0}{8}\times\frac{1}{10^5}\\&=\left(62500+\frac{0}{8}\right)\times\frac{1}{10^5}\\&=62500\times\frac{1}{10^5}\\&=\frac{62500}{10^5}\\&=0.62500\\&=0.625\end{aligned}$$

- Using 10^5 resulted in the same answer. Now we know that we can use any power of 10 with the method of converting a fraction to a decimal.
- This process of selecting a power of 10 to use is similar to putting zeros after the decimal point when we do the long division. You do not quite know how many zeros you will need, and if you put extra that is ok! Using lower powers of 10 can make things more complicated. It is similar to not including enough zeros when doing the long division. For that reason, it is better to use a higher power of 10 because we know the extra zeros will not change the value of the fraction nor its decimal expansion.

Example 2 (5 minutes)

Example 2

Write the decimal expansion of the fraction $\frac{17}{125}$.

- We go through the same process to convert $\frac{17}{125}$ to a finite decimal. We know from Opening Exercise, part (b) that we need to use 10^3 to write $\frac{17}{125}$ as a finite decimal, but from the last example we know that any power of 10 will work.

$$\frac{17}{125} = \frac{17 \times 10^3}{125} \times \frac{1}{10^3}$$

- What do we do next?
  - *Since* $17 \times 10^3 = 17{,}000$*, we need to do division with remainder for* $\frac{17000}{125}$.
- Do the division with remainder and write the next step.
  - $\frac{17000}{125} = 136$, *then* $\frac{17}{125} = \frac{136 \times 125 + 0}{125} \times \frac{1}{10^3}$

Check to make sure all students have the equation above; then instruct them to finish the work and write $\frac{17}{125}$ as a finite decimal.

$$\begin{aligned} &= 136 \times \frac{1}{10^3} \\ &= \frac{136}{10^3} \\ &= 0.136 \end{aligned}$$

Verify that students have the correct decimal; then work on Example 3.

Example 3 (7 minutes)

Example 3

Write the decimal expansion of the fraction $\frac{35}{11}$.

- Now we apply this strategy to a fraction, $\frac{35}{11}$, that is not a finite decimal. How do you know it is not a finite decimal?
  - *We know that the fraction will not be a finite decimal because the denominator is not a product of* 2*'s and/or* 5*'s.*
- What do you think the difference will be in our work?
  - *When we do the division with remainder, we will likely get a remainder, unlike the first two examples that had a remainder of* 0.
- Let's use 10^6 to make sure we get enough decimal digits in order to get a good idea of what the infinite decimal is.

$$\frac{35}{11} = \frac{35 \times 10^6}{11} \times \frac{1}{10^6}$$

What do we do next?

  - *Since* $35 \times 10^6 = 35\,000\,000$, *we need to do division with remainder for* $\frac{35\,000\,000}{11}$.
- We need to determine what numbers make the following statement true.

$35\,000\,000 = \_\_\_\_\_\_\_\_\_\_ \times 11 + \_\_\_\_$.

  - $3{,}181{,}818$ *and* 2 *would give us* $35\,000\,000 = \underline{3\,181\,818} \times 11 + \underline{2}$.
- With this information, we can continue the process.

$$\begin{aligned} \frac{35}{11} &= \frac{3\,181\,818 \times 11 + 2}{11} \times \frac{1}{10^6} \\ &= \left(\frac{3\,181\,818 \times 11}{11} + \frac{2}{11}\right) \times \frac{1}{10^6} \\ &= \left(3\,181\,818 + \frac{2}{11}\right) \times \frac{1}{10^6} \\ &= 3\,181\,818 \times \frac{1}{10^6} + \frac{2}{11} \times \frac{1}{10^6} \\ &= \frac{3\,181\,818}{10^6} + \left(\frac{2}{11} \times \frac{1}{10^6}\right) \\ &= 3.181\,818 + \left(\frac{2}{11} \times \frac{1}{10^6}\right) \end{aligned}$$

At this point we have a fairly good estimation of the decimal expansion of $\frac{35}{11}$ as 3.181818. But we need to consider the value of $\left(\frac{2}{11} \times \frac{1}{10^6}\right)$. We know that $\frac{2}{11} < 1$.

By the basic inequality property, we know that

$$\frac{2}{11} \times \frac{1}{10^6} < 1 \times \frac{1}{10^6}$$
$$\frac{2}{11} \times \frac{1}{10^6} < \frac{1}{10^6}$$

which means that the value of $\frac{2}{11} \times \frac{1}{10^6}$ is less than 0.000001, and we have confirmed that 3.181818 is a good estimation of the infinite decimal that is equal to $\frac{35}{11}$.

Scaffolding:

Consider using a simpler example like $\frac{4}{3}$.

$$\begin{aligned} \frac{4}{3} &= \frac{4 \times 10^2}{3} \times \frac{1}{10^2} \\ &= \frac{133 \times 3 + 1}{3} \times \frac{1}{10^2} \\ &= \left(\frac{133 \times 3}{3} + \frac{1}{3}\right) \times \frac{1}{10^2} \\ &= \left(133 + \frac{1}{3}\right) \times \frac{1}{10^2} \\ &= 133 \times \frac{1}{10^2} + \frac{1}{3} \times \frac{1}{10^2} \\ &= \frac{133}{10^2} + \left(\frac{1}{3} \times \frac{1}{10^2}\right) \\ &= 1.33 + \left(\frac{1}{3} \times \frac{1}{10^2}\right) \end{aligned}$$

Example 4 (8 minutes)

Example 4

Write the decimal expansion of the fraction $\frac{6}{7}$.

- Let's write the decimal expansion of $\frac{6}{7}$. Will it be a finite or infinite decimal? How do you know?
  - *We know that the fraction will not be a finite decimal because the denominator is not a product of* 2*'s and/or* 5*'s.*
- We want to make sure we get enough decimal digits in order to get a good idea of what the infinite decimal is. What power of 10 should we use?
  - *Accept any power of* 10 *students give. Since we know it is an infinite decimal,* 10^6 *should be sufficient to make a good estimate of the value of* $\frac{35}{11}$*, but any power of* 10 *greater than* 6 *will work, too. The work below uses* 10^6*.*
- Using 10^6 we have

$$\frac{6}{7} = \frac{6 \times 10^6}{7} \times \frac{1}{10^6}$$

What do we do next?

  - *Since* $6 \times 10^6 = 6\,000\,000$*, we need to do division with remainder for* $\frac{6\,000\,000}{7}$*.*
- Determine which numbers make the following statement true.

$6\,000\,000 = \underline{\hspace{3cm}} \times 7 + \underline{\hspace{1cm}}$

  - $857{,}142$ *and* 6 *would give us* $6\,000\,000 = \underline{\ }(857\,142) \times 7 + \underline{\ }6$*.*
- Now we know that

$$\frac{6}{7} = \frac{857\,142 \times 7 + 6}{7} \times \frac{1}{10^6}$$

Finish the work to write the decimal expansion of $\frac{6}{7}$.

  - *Sample response:*

$$\begin{aligned}
\frac{6}{7} &= \left(\frac{857\,142 \times 7}{7} + \frac{6}{7}\right) \times \frac{1}{10^6} \\
&= \left(857\,142 + \frac{6}{7}\right) \times \frac{1}{10^6} \\
&= 857\,142 \times \frac{1}{10^6} + \left(\frac{6}{7} \times \frac{1}{10^6}\right) \\
&= \frac{857\,142}{10^6} + \left(\frac{6}{7} \times \frac{1}{10^6}\right) \\
&= 0.857\,142 + \left(\frac{6}{7} \times \frac{1}{10^6}\right)
\end{aligned}$$

- Again, we can verify how good our estimate is using the basic inequality property.

$$\frac{6}{7} < 1$$

$$\frac{6}{7} \times \frac{1}{10^6} < 1 \times \frac{1}{10^6}$$

$$\frac{6}{7} \times \frac{1}{10^6} < \frac{1}{10^6}$$

Therefore, $\frac{6}{7} \times \frac{1}{10^6} < 0.000001$, and stating that $\frac{6}{7} = 0.857142$ is a good estimate.

Exercises 1–3 (5 minutes)

Students complete Exercises 1–3 independently or in pairs. Allow students to use calculators to check their work.

Exercises 1–3

1.

a. Choose a power of 10 to use to convert this fraction to a decimal: $\frac{4}{13}$. Explain your choice.

Choices will vary. The work shown below uses the factor 10^6. Students should choose a factor of at least 10^4 in order to get an approximate decimal expansion and a small remainder that will not greatly affect the value of the number.

b. Determine the decimal expansion of $\frac{4}{13}$. Verify you are correct using a calculator.

$$\begin{aligned}\frac{4}{13} &= \frac{4 \times 10^6}{13} \times \frac{1}{10^6}\\ &= \frac{4\,000\,000}{13} \times \frac{1}{10^6}\end{aligned}$$

$$4\,000\,000 = 307\,692 \times 13 + 4$$

$$\begin{aligned}\frac{4}{13} &= \frac{307\,692 \times 13 + 4}{13} \times \frac{1}{10^6}\\ &= \left(\frac{307\,692 \times 13}{13} + \frac{4}{13}\right) \times \frac{1}{10^6}\\ &= \left(307\,692 + \frac{4}{13}\right) \times \frac{1}{10^6}\\ &= 307\,692 \times \frac{1}{10^6} + \left(\frac{4}{13} \times \frac{1}{10^6}\right)\\ &= \frac{307\,692}{10^6} + \left(\frac{4}{13} \times \frac{1}{10^6}\right)\\ &= 0.307\,692 + \left(\frac{4}{13} \times \frac{1}{10^6}\right)\end{aligned}$$

The decimal expansion of $\frac{4}{13}$ is approximately 0.307692.

2. **Write the decimal expansion of $\frac{1}{11}$. Verify you are correct using a calculator.**

$$\begin{aligned}\frac{1}{11} &= \frac{1\times 10^6}{11}\times\frac{1}{10^6}\\ &= \frac{1\,000\,000}{11}\times\frac{1}{10^6}\end{aligned}$$

$$1\,000\,000 = 90\,909\times 11 + 1$$

$$\begin{aligned}\frac{1}{11} &= \frac{90\,909\times 11 + 1}{11}\times\frac{1}{10^6}\\ &= \left(\frac{90\,909\times 11}{11}+\frac{1}{11}\right)\times\frac{1}{10^6}\\ &= \left(90\,909+\frac{1}{11}\right)\times\frac{1}{10^6}\\ &= 90\,909\times\frac{1}{10^6}+\left(\frac{1}{11}\times\frac{1}{10^6}\right)\\ &= \frac{90\,909}{10^6}+\left(\frac{1}{11}\times\frac{1}{10^6}\right)\\ &= 0.090\,909\,0+\left(\frac{1}{11}\times\frac{1}{10^6}\right)\end{aligned}$$

The decimal expansion of $\frac{1}{11}$ *is approximately* 0.0909090.

3. **Write the decimal expansion of $\frac{19}{21}$. Verify you are correct using a calculator.**

$$\begin{aligned}\frac{19}{21} &= \frac{19\times 10^8}{21}\times\frac{1}{10^8}\\ &= \frac{1\,900\,000\,000}{21}\times\frac{1}{10^8}\end{aligned}$$

$$1\,900\,000\,000 = 90\,476\,190\times 21 + 10$$

$$\begin{aligned}\frac{19}{21} &= \frac{90\,476\,190\times 21 + 10}{21}\times\frac{1}{10^8}\\ &= \left(\frac{90\,476\,190\times 21}{21}+\frac{10}{21}\right)\times\frac{1}{10^8}\\ &= \left(90\,476\,190+\frac{10}{21}\right)\times\frac{1}{10^8}\\ &= 90\,476\,190\times\frac{1}{10^8}+\left(\frac{10}{21}\times\frac{1}{10^8}\right)\\ &= \frac{90\,476\,190}{10^8}+\left(\frac{10}{21}\times\frac{1}{10^8}\right)\\ &= 0.904\,761\,90+\left(\frac{10}{21}\times\frac{1}{10^8}\right)\end{aligned}$$

The decimal expansion of $\frac{19}{21}$ *is approximately* 0.90476190.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know how to write the decimal expansion for any fraction.
- Using what we know about equivalent fractions, we can multiply a fraction by a power of 10 large enough to give us enough decimal digits to estimate the decimal expansion of a fraction.
- We know that the amount we do not include in the decimal expansion is a very small amount that will not change the value of the number in any meaningful way.

Lesson Summary

Multiplying a fraction's numerator and denominator by the same power of 10 to determine its decimal expansion is similar to including extra zeros to the right of a decimal when using the long division algorithm. The method of multiplying by a power of 10 reduces the work to whole number division.

Example: We know that the fraction $\frac{5}{3}$ has an infinite decimal expansion because the denominator is not a product of 2's and/or 5's. Its decimal expansion is found by the following procedure:

$\frac{5}{3} = \frac{5 \times 10^2}{3} \times \frac{1}{10^2}$	**Multiply numerator and denominator by 10^2.**
$= \frac{166 \times 3 + 2}{3} \times \frac{1}{10^2}$	**Rewrite the numerator as a product of a number multiplied by the denominator.**
$= \left(\frac{166 \times 3}{3} + \frac{2}{3}\right) \times \frac{1}{10^2}$	**Rewrite the first term as a sum of fractions with the same denominator.**
$= \left(166 + \frac{2}{3}\right) \times \frac{1}{10^2}$	**Simplify.**
$= \frac{166}{10^2} + \left(\frac{2}{3} \times \frac{1}{10^2}\right)$	**Use the distributive property.**
$= 1.66 + \left(\frac{2}{3} \times \frac{1}{10^2}\right)$	**Simplify.**
$= 166 \times \frac{1}{10^2} + \frac{2}{3} \times \frac{1}{10^2}$	**Simplify the first term using what you know about place value.**

Notice that the value of the remainder, $\left(\frac{2}{3} \times \frac{1}{10^2}\right) = \frac{2}{300} = 0.00\overline{6}$, is quite small and does not add much value to the number. Therefore, 1.66 is a good estimate of the value of the infinite decimal for the fraction $\frac{5}{3}$.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 9: Decimal Expansions of Fractions, Part 1

Exit Ticket

1. Write the decimal expansion of $\frac{823}{40}$.

2. Write the decimal expansion of $\frac{48}{21}$.

Exit Ticket Sample Solutions

1. Write the decimal expansion of $\frac{823}{40}$.

$$\begin{aligned}\frac{823}{40} &= \frac{823 \times 10^3}{40} \times \frac{1}{10^3}\\ &= \frac{823\,000}{40} \times \frac{1}{10^3}\\ &= \frac{20575 \times 40}{40} \times \frac{1}{10^3}\\ &= 20575 \times \frac{1}{10^3}\\ &= \frac{20575}{10^3}\\ &= 20.575\end{aligned}$$

The decimal expansion of $\frac{823}{40}$ *is* 20.575.

2. Write the decimal expansion of $\frac{48}{21}$.

$$\begin{aligned}\frac{48}{21} &= \frac{48 \times 10^6}{21} \times \frac{1}{10^6}\\ &= \frac{48\,000\,000}{21} \times \frac{1}{10^6}\end{aligned}$$

$$48\,000\,000 = 2\,285\,714 \times 21 + 6$$

$$\begin{aligned}\frac{48}{21} &= \frac{2\,285\,714 \times 21 + 6}{21} \times \frac{1}{10^6}\\ &= \left(\frac{2\,285\,714 \times 21}{21} + \frac{6}{21}\right) \times \frac{1}{10^6}\\ &= \left(2\,285\,714 + \frac{6}{21}\right) \times \frac{1}{10^6}\\ &= 2\,285\,714 \times \frac{1}{10^6} + \left(\frac{6}{21} \times \frac{1}{10^6}\right)\\ &= \frac{2\,285\,714}{10^6} + \left(\frac{6}{21} \times \frac{1}{10^6}\right)\\ &= 2.285\,714 + \left(\frac{6}{21} \times \frac{1}{10^6}\right)\end{aligned}$$

The decimal expansion of $\frac{48}{21}$ *is approximately* 2.285714.

Problem Set Sample Solutions

1.

a. Choose a power of 10 to convert this fraction to a decimal: $\frac{4}{11}$. Choices will vary.

The work shown below uses the factor 10^6. *Students should choose a factor of at least* 10^4 *in order to get an approximate decimal expansion and notice that the decimal expansion repeats.*

b. Determine the decimal expansion of $\frac{4}{11}$. Verify you are correct using a calculator.

$$\frac{4}{11}=\frac{4\times 10^6}{11}\times\frac{1}{10^6}$$
$$=\frac{4\ 000\ 000}{11}\times\frac{1}{10^6}$$

$$4\ 000\ 000 = 363\ 636\times 11+4$$

$$\frac{4}{11}=\frac{363\ 636\times 11+4}{11}\times\frac{1}{10^6}$$
$$=\left(\frac{363\ 636\times 11}{11}+\frac{4}{11}\right)\times\frac{1}{10^6}$$
$$=\left(363\ 636+\frac{4}{11}\right)\times\frac{1}{10^6}$$
$$=363\ 636\times\frac{1}{10^6}+\left(\frac{4}{11}\times\frac{1}{10^6}\right)$$
$$=\frac{363\ 636}{10^6}+\left(\frac{4}{11}\times\frac{1}{10^6}\right)$$
$$=0.363\ 636+\left(\frac{4}{11}\times\frac{1}{10^6}\right)$$

The decimal expansion of $\frac{4}{11}$ *is approximately* 0.363636.

2. Write the decimal expansion of $\frac{5}{13}$. Verify you are correct using a calculator.

$$\frac{5}{13}=\frac{5\times 10^6}{13}\times\frac{1}{10^6}$$
$$=\frac{5\ 000\ 000}{13}\times\frac{1}{10^6}$$

$$5\ 000\ 000 = 384\ 615\times 13+5$$

$$\frac{5}{13}=\frac{384\ 615\times 13+5}{13}\times\frac{1}{10^6}$$
$$=\left(\frac{384\ 615\times 13}{13}+\frac{5}{13}\right)\times\frac{1}{10^6}$$
$$=\left(384\ 615+\frac{5}{13}\right)\times\frac{1}{10^6}$$
$$=384\ 615\times\frac{1}{10^6}+\left(\frac{5}{13}\times\frac{1}{10^6}\right)$$
$$=\frac{384\ 615}{10^6}+\left(\frac{5}{13}\times\frac{1}{10^6}\right)$$
$$=0.384\ 615+\left(\frac{5}{13}\times\frac{1}{10^6}\right)$$

The decimal expansion of $\frac{5}{13}$ *is approximately* 0.384615.

3. Write the decimal expansion of $\frac{23}{39}$. Verify you are correct using a calculator.

$$\frac{23}{39}=\frac{23\times 10^6}{39}\times\frac{1}{10^6}$$
$$=\frac{23\ 000\ 000}{39}\times\frac{1}{10^6}$$

$$23\ 000\ 000 = 589\ 743\times 39+23$$

$$\frac{23}{39}=\frac{589\ 743\times 39+23}{39}\times\frac{1}{10^6}$$
$$=\left(\frac{589\ 743\times 39}{39}+\frac{23}{39}\right)\times\frac{1}{10^6}$$
$$=\left(589\ 743+\frac{23}{39}\right)\times\frac{1}{10^6}$$
$$=589\ 743\times\frac{1}{10^6}+\left(\frac{23}{39}\times\frac{1}{10^6}\right)$$
$$=\frac{589\ 743}{10^6}+\left(\frac{23}{39}\times\frac{1}{10^6}\right)$$
$$=0.589\ 743+\left(\frac{23}{39}\times\frac{1}{10^6}\right)$$

The decimal expansion of $\frac{23}{39}$ *is approximately* 0.589743.

4. **Tamer wrote the decimal expansion of $\frac{3}{7}$ as 0.418571, but when he checked it on a calculator, it was 0.428571. Identify his error and explain what he did wrong.**

$$\begin{aligned}\frac{3}{7} &= \frac{3 \times 10^6}{7} \times \frac{1}{10^6}\\ &= \frac{3\,000\,000}{7} \times \frac{1}{10^6}\end{aligned}$$

$$3\,000\,000 = 418\,571 \times 7 + 3$$

$$\begin{aligned}\frac{3}{7} &= \frac{418\,571 \times 7 + 3}{7} \times \frac{1}{10^6}\\ &= \left(\frac{418\,571 \times 7}{7} + \frac{3}{7}\right) \times \frac{1}{10^6}\\ &= \left(418\,571 + \frac{3}{7}\right) \times \frac{1}{10^6}\\ &= 418\,571 \times \frac{1}{10^6} + \left(\frac{3}{7} \times \frac{1}{10^6}\right)\\ &= \frac{418\,571}{10^6} + \left(\frac{3}{7} \times \frac{1}{10^6}\right)\\ &= 0.418\,571 + \left(\frac{3}{7} \times \frac{1}{10^6}\right)\end{aligned}$$

Tamer did the division with remainder incorrectly. He wrote that $3\,000\,000 = 418\,571 \times 7 + 3$ when actually $3\,000\,000 = 428\,571 \times 7 + 3$. This error led to his decimal expansion being incorrect.

5. **Given that $\frac{6}{7} = 0.857\,142 + \left(\frac{6}{7} \times \frac{1}{10^6}\right)$, explain why 0.857142 is a good estimate of $\frac{6}{7}$.**

When you consider the value of $\left(\frac{6}{7} \times \frac{1}{10^6}\right)$, then it is clear that 0.857142 is a good estimate of $\frac{6}{7}$. We know that $\frac{6}{7} < 1$. By the basic inequality property, we also know that $\frac{6}{7} \times \frac{1}{10^6} < 1 \times \frac{1}{10^6}$, which means that $\frac{6}{7} \times \frac{1}{10^6} < 0.000001$. That is such a small value that it will not affect the estimate of $\frac{6}{7}$ in any real way.

Lesson 10: Converting Repeating Decimals to Fractions

Student Outcomes

- Students know intuitively that every repeating decimal is equal to a fraction. Students convert a decimal that eventually repeats into a fraction.
- Students know that the decimal expansions of rational numbers repeat eventually.
- Students understand that irrational numbers are numbers that are not rational. Irrational numbers cannot be represented as a fraction and have infinite decimals that never repeat.

Classwork

Discussion (4 minutes)

- We have just seen that every fraction (therefore every rational number) is equal to a repeating decimal, and we have learned strategies for determining the decimal expansion of fractions. Now we must learn how to write a repeating decimal as a fraction.
- We begin by noting a simple fact about finite decimals: Given a finite decimal, such as 1.2345678, if we multiply the decimal by 10^5, we get $123{,}456.78$. That is, when we multiply by a power of 10, in this case 10^5, the decimal point is moved 5 places to the right, i.e.,

  $$1.2345678 \times 10^5 = 123\,456.78$$

  This is true because of what we know about the laws of exponents:

  $$\begin{aligned} 10^5 \times 1.2\,345\,678 &= 10^5 \times (12\,345{,}678 \times 10^{-7}) \\ &= 12\,345\,678 \times 10^{-2} \\ &= 123\,456.78 \end{aligned}$$

- We have discussed in previous lessons that we treat infinite decimals as finite decimals in order to compute with them. For that reason, we will now apply the same basic fact we observed about finite decimals to infinite decimals. That is,

  $$1.2345678 \ldots \times 10^5 = 123\,456.78 \ldots.$$

  We will use this fact to help us write infinite decimals as fractions.

Example 1 (10 minutes)

Example 1

Find the fraction that is equal to the infinite decimal $0.\overline{81}$.

- We want to find the fraction that is equal to the infinite decimal $0.\overline{81}$. Why might we want to write an infinite decimal as a fraction?

Provide time for students to discuss a reason for wanting to do this. In previous lessons, we said that infinite decimals must be approximated in order to compute with them. When we compute with approximations, our answers are not precise. If we can rewrite an infinite decimal as a fraction, we can more easily compute, thus leading to a more precise answer. It may be necessary to simply tell them, but let them think first.

- For the stated reason, we must figure out how to find the fraction that is equal to the infinite decimal $0.\overline{81}$.
- We let $x = 0.\overline{81}$.

MP.1

Allow students time to work in pairs or small groups to write the fraction equal to $0.\overline{81}$. Students should recognize that the preceding discussion has something to do with this process and should be an entry point for finding the solution. They should also recognize that since we let $x = 0.\overline{81}$, an equation of some form will lead them to the fraction. Give them time to make sense of the problem. Make a plan for finding the fraction, and then attempt to figure it out.

- Since $x = 0.\overline{81}$, we will multiply both sides of the equation by 10^2 and then solve for x. We will multiply by 10^2 because there are two decimal digits that repeat immediately following the decimal point.

$$\begin{aligned} x &= 0.\overline{81} \\ x &= 0.818\,181\,81\ldots \\ 10^2x &= (10^2)0.818\,181\,81\ldots \\ 100x &= 81.818\,181\ldots \end{aligned}$$

  Ordinarily, we would finish solving for x by dividing both sides of the equation by 100. Do you see why that is not a good plan for this problem?

  - *If we divide both sides by* 100*, we would get* $x = \frac{81.818181\ldots}{100}$*, which does not really show us that the repeating decimal is equal to a fraction (rational number) because the repeating decimal is still in the numerator.*

MP.7

- We know that $81.818181\ldots$ is the same as $81 + 0.818181\ldots$. Then by substitution, we have $100x = 81 + 0.818181\ldots$.

  How can we rewrite $100x = 81 + 0.818181\ldots$ in a useful way using the fact that $x = 0.\overline{81}$?

  - *We can rewrite* $100x = 81 + 0.818181\ldots$ *as* $100x = 81 + x$ *because* x *represents the repeating decimal block* $0.818181\ldots$*.*

  Now we can solve for x to find the fraction that represents the repeating decimal $0.\overline{81}$:

$$\begin{aligned} 100x &= 81 + x \\ 100x - x &= 81 + x - x \\ (100 - 1)x &= 81 \\ 99x &= 81 \\ \frac{99x}{99} &= \frac{81}{99} \\ x &= \frac{81}{99} \\ x &= \frac{9}{11} \end{aligned}$$

- Therefore, the repeating decimal $0.\overline{81} = \frac{9}{11}$.

Have students use calculators to verify that we are correct.

Exercises 1–2 (5 minutes)

Students complete Exercises 1–2 in pairs. Allow them to use calculators to check their work.

Exercises 1–2

1.

a. **Let $x = 0.\overline{123}$. Explain why multiplying both sides of this equation by 10^3 will help us determine the fractional representation of x.**

When we multiply both sides of the equation by 10^3, on the right side we will have 123.123123 …. This is helpful because we will be able to subtract the repeating decimal from both sides by subtracting x.

b. **After multiplying both sides of the equation by 10^3, rewrite the resulting equation by making a substitution that will help determine the fractional value of x. Explain how you were able to make the substitution.**

$$\begin{aligned} x &= 0.\overline{123} \\ 10^3x &= (10^3)0.\overline{123} \\ 1000x &= 123.\overline{123} \\ 1000x &= 123 + 0.123\,123 \ldots \\ 1000x &= 123 + x \end{aligned}$$

Since we let $x = 0.\overline{123}$, we can substitute the repeating decimal 0.123123 … with x.

c. **Solve the equation to determine the value of x.**

$$\begin{aligned} 1000x - x &= 123 + x - x \\ 999x &= 123 \\ \frac{999x}{999} &= \frac{123}{999} \\ x &= \frac{123}{999} \\ x &= \frac{41}{333} \end{aligned}$$

d. **Is your answer reasonable? Check your answer using a calculator.**

Yes, my answer is reasonable and correct. It is reasonable because the denominator cannot be expressed as a product of 2's and 5's; therefore, I know that the fraction must represent an infinite decimal. It is also reasonable because the decimal value is closer to 0 than to 0.5, and the fraction $\frac{41}{333}$ is also closer to 0 than to $\frac{1}{2}$. It is correct because the division of $\frac{41}{333}$ using a calculator is 0.123123 ….

2. **Find the fraction equal to $0.\overline{4}$. Check your answer using a calculator.**

Let $x = 0.\overline{4}$

$$\begin{aligned} x &= 0.\overline{4} \\ 10x &= (10)0.\overline{4} \\ 10x &= 4.\overline{4} \\ 10x &= 4 + x \\ 10x - x &= 4 + x - x \\ 9x &= 4 \\ \frac{9x}{9} &= \frac{4}{9} \\ x &= \frac{4}{9} \end{aligned}$$

Example 2 (6 minutes)

Example 2

Find the fraction that is equal to the infinite decimal $2.13\overline{8}$.

- We want to find the fraction that is equal to the infinite decimal $2.13\overline{8}$. Notice that this time there is just one digit that repeats, but it is three places to the right of the decimal point. If we let $x = 2.13\overline{8}$, by what power of 10 should we multiply? Explain.
    - *The goal is to multiply by a power of* 10 *so that the only remaining decimal digits are those that repeat. For that reason, we should multiply by* 10^2.
- We let $x = 2.13\overline{8}$, and multiply both sides of the equation by 10^2.

$$\begin{aligned} x &= 2.13\overline{8} \\ 10^2x &= (10^2)2.13\overline{8} \\ 100x &= 213.\overline{8} \\ 100x &= 213 + 0.\overline{8} \end{aligned}$$

This time, we cannot simply subtract x from each side. Explain why.

- *Subtracting* x *in previous problems allowed us to completely remove the repeating decimal. This time,* $x = 2.13\overline{8}$*, not just* $0.\overline{8}$.

- What we will do now is treat $0.\overline{8}$ as a separate, mini-problem. Determine the fraction that is equal to $0.\overline{8}$.
    - *Let* $y = 0.\overline{8}$.

$$\begin{aligned} y &= 0.\overline{8} \\ 10y &= 8.\overline{8} \\ 10y &= 8 + 0.\overline{8} \\ 10y &= 8 + y \\ 10y - y &= 8 + y - y \\ 9y &= 8 \\ \frac{9y}{9} &= \frac{8}{9} \\ y &= \frac{8}{9} \end{aligned}$$

- Now that we know that $0.\overline{8}=\frac{8}{9}$, we will go back to our original problem.

$$\begin{aligned}
100x &= 213+0.\overline{8}\\
100x &= 213+\frac{8}{9}\\
100x &= \frac{213\times 9}{9}+\frac{8}{9}\\
100x &= \frac{213\times 9+8}{9}\\
100x &= \frac{1925}{9}\\
\frac{1}{100}(100x) &= \frac{1925}{9}\left(\frac{1}{100}\right)\\
x &= \frac{1925}{900}\\
x &= \frac{77}{36}
\end{aligned}$$

Exercises 3–4 (6 minutes)

Students complete Exercises 3–4 independently or in pairs. Allow students to use calculators to check their work.

Exercises 3–4

3. Find the fraction equal to $1.6\overline{23}$. Check your answer using a calculator.

Let $x=1.6\overline{23}$

$$\begin{aligned}
x &= 1.6\overline{23}\\
10x &= (10)1.6\overline{23}\\
10x &= 16.\overline{23}
\end{aligned}$$

Let $y=0.\overline{23}$

$$\begin{aligned}
y &= 0.\overline{23}\\
10^2y &= (10^2)0.\overline{23}\\
100y &= 23.\overline{23}\\
100y &= 23+y\\
100y-y &= 23+y-y\\
99y &= 23\\
\frac{99y}{99} &= \frac{23}{99}\\
y &= \frac{23}{99}
\end{aligned}$$

$$\begin{aligned}
10x &= 16.\overline{23}\\
10x &= 16+y\\
10x &= 16+\frac{23}{99}\\
10x &= \frac{16\times 99}{99}+\frac{23}{99}\\
10x &= \frac{16\times 99+23}{99}\\
10x &= \frac{1607}{99}\\
\frac{1}{10}(10x) &= \frac{1}{10}\left(\frac{1607}{99}\right)\\
x &= \frac{1607}{990}
\end{aligned}$$

$$1.6\overline{23}=\frac{1607}{990}$$

4. **Find the fraction equal to $2.9\overline{60}$. Check your answer using a calculator.**

Let $x = 2.9\overline{60}$

$$x = 2.9\overline{60}$$
$$10x = (10)2.9\overline{60}$$
$$10x = 29.\overline{60}$$

Let $y = 0.\overline{60}$

$$y = 0.\overline{60}$$
$$10^2y = (10^2)0.\overline{60}$$
$$100y = 60.\overline{60}$$
$$100y = 60 + y$$
$$100y - y = 60 + y - y$$
$$99y = 60$$
$$\frac{99y}{99} = \frac{60}{99}$$
$$y = \frac{60}{99}$$
$$y = \frac{20}{33}$$

$$10x = 29.\overline{60}$$
$$10x = 29 + y$$
$$10x = 29 + \frac{20}{33}$$
$$10x = \frac{29 \times 33}{33} + \frac{20}{33}$$
$$10x = \frac{29 \times 33 + 20}{33}$$
$$10x = \frac{977}{33}$$
$$\left(\frac{1}{10}\right)10x = \left(\frac{1}{10}\right)\frac{977}{33}$$
$$x = \frac{977}{330}$$

$$2.9\overline{60} = \frac{977}{330}$$

Discussion (4 minutes)

- What we have observed so far is that when an infinite decimal repeats, it can be written as a fraction, which means that it is a rational number. Do you think infinite decimals that do not repeat are rational as well? Explain.

Provide students time to discuss with a partner before sharing their thoughts with the class.

  - *Considering the work from this lesson, it does not seem reasonable that an infinite decimal that does not repeat can be expressed as a fraction. We would not have a value that we could set for x and use to compute in order to find the fraction. For those reasons, we do not believe that an infinite decimal that does not repeat is a rational number.*

- Infinite decimals that do not repeat are *irrational numbers*; in other words, when a number is not equal to a rational number, it is irrational. What we will learn next is how to use rational approximation to determine the approximate decimal expansion of an irrational number.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that to work with infinite decimals we must treat them as finite decimals.
- We know how to use our knowledge of powers of 10 and linear equations to write an infinite decimal that repeats as a fraction.
- We know that every decimal that eventually repeats is a rational number.

Lesson Summary

Numbers with decimal expansions that repeat are rational numbers and can be converted to fractions using a linear equation.

Example: Find the fraction that is equal to the number $0.\overline{567}$.

Let x represent the infinite decimal $0.\overline{567}$.

$x = 0.\overline{567}$	
$10^3x = 10^3(0.\overline{567})$	**Multiply by 10^3 because there are 3 digits that repeat.**
$1000x = 567.\overline{567}$	**Simplify.**
$1000x = 567 + 0.\overline{567}$	**By addition**
$1000x = 567 + x$	**By substitution; $x = 0.\overline{567}$**
$1000x - x = 567 + x - x$	**Subtraction property of equality**
$999x = 567$	**Simplify.**
$\frac{999}{999}x = \frac{567}{999}$	**Division property of equality**
$x = \frac{567}{999} = \frac{63}{111}$	**Simplify.**

This process may need to be used more than once when the repeating digits, as in numbers such as $1.2\overline{6}$, do not begin immediately after the decimal.

Irrational numbers are numbers that are not rational. They have infinite decimals that do not repeat and cannot be represented as a fraction.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 10: Converting Repeating Decimals to Fractions

Exit Ticket

1. Find the fraction equal to $0.\overline{534}$.

2. Find the fraction equal to $3.0\overline{15}$.

Exit Ticket Sample Solutions

1. **Find the fraction equal to $0.\overline{534}$.**

Let $x = 0.\overline{534}$.

$$\begin{aligned} x &= 0.\overline{534} \\ 10^3x &= (10^3)0.\overline{534} \\ 1000x &= 534.\overline{534} \\ 1000x &= 534 + x \\ 1000x - x &= 534 + x - x \\ 999x &= 534 \\ \frac{999x}{999} &= \frac{534}{999} \\ x &= \frac{534}{999} \\ x &= \frac{178}{333} \end{aligned}$$

$0.\overline{534} = \frac{178}{333}$

2. **Find the fraction equal to $3.0\overline{15}$.**

Let $x = 3.0\overline{15}$

$$\begin{aligned} x &= 3.0\overline{15} \\ 10x &= (10)3.0\overline{15} \\ 10x &= 30.\overline{15} \end{aligned}$$

Let $y = 0.\overline{15}$

$$\begin{aligned} y &= 0.\overline{15} \\ 10^2y &= (10^2)0.\overline{15} \\ 100y &= 15.\overline{15} \\ 100y &= 15 + y \\ 100y - y &= 15 + y - y \\ 99y &= 15 \\ \frac{99y}{99} &= \frac{15}{99} \\ y &= \frac{5}{33} \end{aligned}$$

$$\begin{aligned} 10x &= 30.\overline{15} \\ 10x &= 30 + y \\ 10x &= 30 + \frac{5}{33} \\ 10x &= \frac{30 \times 33}{33} + \frac{5}{33} \\ 10x &= \frac{30 \times 33 + 5}{33} \\ 10x &= \frac{995}{33} \\ \frac{1}{10}(10x) &= \frac{1}{10}\left(\frac{995}{33}\right) \\ x &= \frac{995}{330} \\ x &= \frac{199}{66} \end{aligned}$$

$3.0\overline{15} = \frac{199}{66}$

Problem Set Sample Solutions

1.

a. **Let $x = 0.\overline{631}$. Explain why multiplying both sides of this equation by 10^3 will help us determine the fractional representation of x.**

When we multiply both sides of the equation by 10^3, on the right side we will have $631.631631\ldots$. This is helpful because we will be able to subtract the repeating decimal from both sides by subtracting x.

b. **After multiplying both sides of the equation by 10^3, rewrite the resulting equation by making a substitution that will help determine the fractional value of x. Explain how you were able to make the substitution.**

$$x = 0.\overline{631}$$
$$10^3x = (10^3)0.\overline{631}$$
$$1000x = 631.\overline{631}$$
$$1000x = 631 + 0.631\,631\ldots$$
$$1000x = 631 + x$$

Since we let $x = 0.\overline{631}$, we can substitute the repeating decimal $0.631631\ldots$ with x.

c. **Solve the equation to determine the value of x.**

$$1000x - x = 631 + x - x$$
$$999x = 631$$
$$\frac{999x}{999} = \frac{631}{999}$$
$$x = \frac{631}{999}$$

d. **Is your answer reasonable? Check your answer using a calculator.**

Yes, my answer is reasonable and correct. It is reasonable because the denominator cannot be expressed as a product of 2's and 5's; therefore, I know that the fraction must represent an infinite decimal. Also, the number 0.631 is closer to 0.5 than 1, and the fraction is also closer to $\frac{1}{2}$ than 1. It is correct because the division of $\frac{631}{999}$ using the calculator is $0.631631\ldots$.

2. **Find the fraction equal to $3.40\overline{\overline{8}}$. Check your answer using a calculator.**

Let $x = 3.40\overline{8}$

$$x = 3.40\overline{8}$$
$$10^2x = (10^2)3.40\overline{8}$$
$$100x = 340.\overline{8}$$

Let $y = 0.\overline{8}$

$$y = 0.\overline{8}$$
$$10y = 10(0.\overline{8})$$
$$10y = 8.\overline{8}$$
$$10y = 8 + y$$
$$10y - y = 8 + y - y$$
$$9y = 8$$
$$\frac{9y}{9} = \frac{8}{9}$$
$$y = \frac{8}{9}$$

$$100x = 340.\overline{8}$$
$$100x = 340 + y$$
$$100x = 340 + \frac{8}{9}$$
$$100x = \frac{340 \times 9}{9} + \frac{8}{9}$$
$$100x = \frac{340 \times 9 + 8}{9}$$
$$100x = \frac{3068}{9}$$
$$\left(\frac{1}{100}\right)100x = \left(\frac{1}{100}\right)\frac{3068}{9}$$
$$x = \frac{3068}{900}$$
$$x = \frac{767}{225}$$

$$3.40\overline{8} = \frac{767}{225}$$

3. **Find the fraction equal to $0.\overline{5923}$. Check your answer using a calculator.**

Let $x = 0.\overline{5923}$

$$x = 0.\overline{5923}$$
$$10^4x = (10^4)0.\overline{5923}$$
$$10\,000x = 5923.\overline{5923}$$
$$10\,000x = 5923 + x$$
$$10\,000x - x = 5923 + x - x$$
$$9999x = 5923$$
$$\frac{9999x}{9999} = \frac{5923}{9999}$$
$$x = \frac{5923}{9999}$$

4. **Find the fraction equal to $2.3\overline{82}$. Check your answer using a calculator.**

Let $x = 2.3\overline{82}$

$$x = 2.3\overline{82}$$
$$10x = (10)2.3\overline{82}$$
$$10x = 23.\overline{82}$$

Let $y = 0.\overline{82}$

$$y = 0.\overline{82}$$
$$10^2y = (10^2)0.\overline{82}$$
$$100y = 82.\overline{82}$$
$$100y = 82 + y$$
$$100y - y = 82 + y - y$$
$$99y = 82$$
$$\frac{99y}{99} = \frac{82}{99}$$
$$y = \frac{82}{99}$$

$$10x = 23.\overline{82}$$
$$10x = 23 + \frac{82}{99}$$
$$10x = \frac{23 \times 99}{99} + \frac{82}{99}$$
$$10x = \frac{23 \times 99 + 82}{99}$$
$$10x = \frac{2359}{99}$$
$$\left(\frac{1}{10}\right)10x = \left(\frac{1}{10}\right)\frac{2359}{99}$$
$$x = \frac{2359}{990}$$

$$2.3\overline{82} = \frac{2359}{990}$$

5. **Find the fraction equal to $0.\overline{714285}$. Check your answer using a calculator.**

Let $x = 0.\overline{714285}$.

$$\begin{aligned} x &= 0.\overline{714\,285} \\ 10^6x &= (10^6)0.\overline{714\,285} \\ 1\,000\,000x &= 714\,825.\overline{714\,285} \\ 1\,000\,000x &= 714\,285 + x \\ 1\,000\,000x - x &= 714\,285 + x - x \\ 999\,999x &= 714\,285 \\ \frac{999\,999x}{999\,999} &= \frac{714\,285}{999\,999} \\ x &= \frac{714\,285}{999\,999} \\ x &= \frac{5}{7} \end{aligned}$$

6. **Explain why an infinite decimal that is not a repeating decimal cannot be rational.**

Infinite decimals that do repeat can be expressed as a fraction and are therefore rational. The method we learned today for writing a repeating decimal as a rational number cannot be applied to infinite decimals that do not repeat. The method requires that we let x represent the repeating part of the decimal. If the number has a decimal expansion that does not repeat, we cannot express the number as a fraction (i.e., a rational number).

7. **In a previous lesson, we were convinced that it is acceptable to write $0.\overline{9} = 1$. Use what you learned today to show that it is true.**

Let $x = 0.\overline{9}$

$$\begin{aligned} x &= 0.\overline{9} \\ 10x &= (10)0.\overline{9} \\ 10x &= 9.\overline{9} \\ 10x &= 9 + x \\ 10x - x &= 9 + x - x \\ 9x &= 9 \\ \frac{9x}{9} &= \frac{9}{9} \\ x &= \frac{9}{9} \\ x &= 1 \end{aligned}$$

8. **Examine the following repeating infinite decimals and their fraction equivalents. What do you notice? Why do you think what you observed is true?**

$$0.\overline{81} = \frac{81}{99} \qquad 0.\overline{4} = \frac{4}{9} \qquad 0.\overline{123} = \frac{123}{999} \qquad 0.\overline{60} = \frac{60}{99} \qquad 0.\overline{9} = 1.0$$

In each case, the fraction that represents the infinite decimal has a numerator that is exactly the repeating part of the decimal and a denominator comprised of 9's. Specifically, the denominator has the same number of digits of 9's as the number of digits that repeat. For example, $0.\overline{81}$ has two repeating decimal digits, so the denominator has two 9's. Since we know that $0.\overline{9} = 1$, we can make the assumption that repeating 9's, like 99, could be expressed as 100, meaning that the fraction $\frac{81}{99}$ is almost $\frac{81}{100}$, which would then be expressed as 0.81.

Lesson 11: The Decimal Expansion of Some Irrational Numbers

Student Outcomes

- Students use rational approximation to get the approximate decimal expansion of numbers like $\sqrt{3}$ and $\sqrt{28}$.
- Students distinguish between rational and irrational numbers based on decimal expansions.

Lesson Notes

The definition of an irrational number can finally be given and understood completely once students know that the decimal expansion of non-perfect squares like $\sqrt{3}$ and $\sqrt{28}$ are infinite and do not repeat. That is, square roots of non-perfect squares cannot be expressed as rational numbers and are therefore defined as irrational numbers.

Classwork

Opening Exercise (5 minutes)

Opening Exercise

Place $\sqrt{28}$ on a number line. What decimal do you think $\sqrt{28}$ is equal to? Explain your reasoning.

MP.3 Lead a discussion where students share their reasoning as to the placement of $\sqrt{28}$ on the number line. Encourage students to critique the reasoning of others while evaluating their own arguments. Consider having students vote on the placement they think is most correct.

Discussion (10 minutes)

- We have studied the properties of rational numbers; today, we will finally be able to characterize those numbers that are not rational.
- So far, we have been able to estimate the size of a number like $\sqrt{3}$ by stating that it is between the two perfect squares $\sqrt{1}$ and $\sqrt{4}$, meaning that $\sqrt{3}$ is between 1 and 2 but closer to 2. In our work so far, we have found the decimal expansion of numbers by using long division and by inspecting the denominators for products of 2's and 5's. Numbers written with a square root symbol are different and require a different method for determining their decimal expansions. The method we will learn is called rational approximation: using a sequence of rational numbers to get closer and closer to a given number to estimate the value of a number.

Example 1

Example 1

Recall the basic theorem on inequalities:

Let c and d be two positive numbers, and let n be a fixed positive integer. Then $c < d$ if and only if $c^n < d^n$.

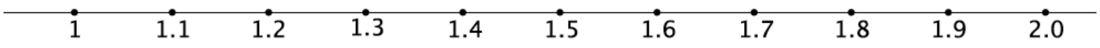
Write the decimal expansion of $\sqrt{3}$.

First approximation:

- We will use the basic theorem on inequalities that we learned in Lesson 3:

  Let c and d be two positive numbers, and let n be a fixed positive integer. Then $c < d$ if and only if $c^n < d^n$.
- To write the decimal expansion of $\sqrt{3}$, we first determine between which two integers the number $\sqrt{3}$ would lie on the number line. This is our first approximation. What are those integers?
  - *The number $\sqrt{3}$ will be between 1 and 2 on the number line because $1^2 = 1$ and $2^2 = 4$.*
- With respect to the basic inequality, we can verify that $\sqrt{3}$ lies between the integers 1 and 2 because $1^2 < \left(\sqrt{3}\right)^2 < 2^2$.
- To be more precise with our estimate of $\sqrt{3}$, we now look at the tenths between the numbers 1 and 2. This is our second approximation.

Second approximation:

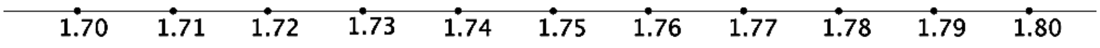


- The question becomes, where exactly would $\sqrt{3}$ lie on this magnified version of the number line? There are 10 possibilities: $1.0 < \sqrt{3} < 1.1$, $1.1 < \sqrt{3} < 1.2$, $1.2 < \sqrt{3} < 1.3$, …, or $1.9 < \sqrt{3} < 2.0$. Use of the basic inequality can guide us to selecting the correct possibility. Specifically, we need to determine which of the inequalities shown below is correct:

  $$1.0^2 < \left(\sqrt{3}\right)^2 < 1.1^2,\ 1.1^2 < \left(\sqrt{3}\right)^2 < 1.2^2,\ 1.2^2 < \left(\sqrt{3}\right)^2 < 1.3^2,\ \ldots, \text{ or } 1.9^2 < \left(\sqrt{3}\right)^2 < 2.0^2.$$

  With the help of a calculator, we can see that $1.7^2 < \left(\sqrt{3}\right)^2 < 1.8^2$ because $1.7^2 = 2.89$ and $1.8^2 = 3.24$; therefore, $1.7 < \sqrt{3} < 1.8$.
- What do you think will need to be done to get an even more precise estimate of the number $\sqrt{3}$?
  - *We will need to look at the interval between 1.7 and 1.8 more closely and repeat the process we did before.*
- Looking at the increments between 1.7 and 1.8, we again have 10 possibilities. This is our third approximation.

Third approximation:



- Using the basic theorem on inequalities and the help of a calculator, show that $\sqrt{3}$ will be between 1.73 and 1.74 because $1.73^2 < \left(\sqrt{3}\right)^2 < 1.74^2$.

Have students verify using a calculator that $1.73^2 = 2.9929$ and $1.74^2 = 3.0276$ and ultimately that $1.73^2 < \left(\sqrt{3}\right)^2 < 1.74^2$.

- What do you think will need to be done to get an even more precise estimate of the number $\sqrt{3}$?
    - *We will need to look at the interval between* 1.73 *and* 1.74 *more closely and repeat the process we did before.*
- At this point the pattern should be clear. Now let's look more carefully at what we are actually doing. We began by looking at the sequence of integers, specifically between two positive integers 1 and 2. Think of this interval as 10^0 (because it equals 1). Then we looked at the sequence of tenths between 1 and 2; think of this interval as 10^{-1} (because it equals $\frac{1}{10}$). Then we looked at the sequence of hundredths between 1.7 and 1.8; think of this interval as 10^{-2} (because it equals $\frac{1}{100}$). To determine the location of $\sqrt{3}$, we had to look between points that differ by 10^{-n} for any positive integer n. The intervals we investigated (i.e., 10^{-n}), get increasingly smaller as n gets larger.
- This method of looking at successive intervals is what we call rational approximation. With each new interval we are approximating the value of the number by determining which two rational numbers it lies between.

Scaffolding:

Defining the terms *approximate*, *approximately*, and *approximation* may be useful to English language learners.

Discussion (15 minutes)

The following discussion revisits the Opening Exercise. Before you begin, ask students to reevaluate their own reasoning, and if you had them vote, consider asking them to vote again to see if anyone wants to change their mind about the best approximation for $\sqrt{28}$.

Example 2

Example 2

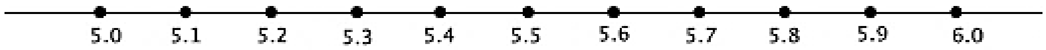
Write the decimal expansion of $\sqrt{28}$.

First approximation:

- We will use the method of rational approximation to estimate the location of $\sqrt{28}$ on the number line.
- What interval of integers (i.e., an interval equal to 10^0), do we examine first? Explain.
    - *We must examine the interval between* 5 *and* 6 *because* $5^2 < \left(\sqrt{28}\right)^2 < 6^2$*; in other words,* $25 < 28 < 36$.

- Now we examine the interval of tenths; in other words, 10^{-1}, between 5 and 6. Where might $\sqrt{28}$ lie?

Second approximation:

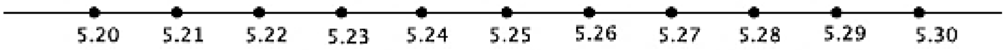


    - *The number* $\sqrt{28}$ *will lie between* 5.0 *and* 5.1*, or* 5.1 *and* 5.2*,…, or* 5.9 *and* 6.0*.*
- How do we determine which interval is correct?
    - *We must use the basic inequality to check each interval. For example, we need to see if the following inequality is true:* $5.0^2 < \left(\sqrt{28}\right)^2 < 5.1^2$.
- Before we begin checking each interval, let's think about how we can be more methodical in our approach. We know that $\sqrt{28}$ is between 5 and 6, but which integer is it closer to?
    - *The number* $\sqrt{28}$ *will be closer to* 5 *than* 6*.*
- Then we should begin checking the intervals beginning with 5 and work our way up. If the number were closer to 6, then we would begin checking the intervals on the right first and work our way down.

Provide students time to determine which interval the number $\sqrt{28}$ will lie between. Ask students to share their findings, and then continue the discussion.

- Now that we know that the number $\sqrt{28}$ lies between 5.2 and 5.3, let's check intervals of hundredths, that is, 10^{-2}.

Third approximation:



- Again, we should try to be methodical. Since $5.2^2 = 27.04$ and $5.3^2 = 28.09$, where should we begin checking?
    - *We should begin with the interval between* 5.29 *and* 5.30 *because* 28 *is closer to* 28.09 *compared to* 27.04*.*

Provide students time to determine which interval the number $\sqrt{28}$ will lie between. Ask students to share their findings, and then continue the discussion.

- Now we know that the number $\sqrt{28}$ is between 5.29 and 5.30. Let's go one step further and examine intervals of thousandths, that is, 10^{-3}.

Fourth approximation:

- Since $5.29^2 = 27.9841$ and $5.30^2 = 28.09$, where should we begin our search?
    - *We should begin with the interval between* 5.290 *and* 5.291 *because* 28 *is closer to* 27.9841 *compared to* 28.09*.*

Provide students time to determine which interval the number $\sqrt{28}$ will lie between. Ask students to share their findings, and then finish the discussion.

- The number $\sqrt{28}$ lies between 5.291 and 5.292 because $5.291^2 = 27.994681$ and $5.292^2 = 28.005264$. At this point, we have a fair approximation of the value of $\sqrt{28}$. It is between 5.291 and 5.292 on the number line:

- We could continue this process of rational approximation to see that $\sqrt{28} = 5.291502622 \ldots$. How is this number different from other infinite decimals we have worked with?
  - *Other infinite decimals we have worked with have a block of digits that repeat at some point. This infinite decimal does not seem to do that.*
- We know that rational numbers are those that have decimal expansions that eventually repeat. We also know that a rational number can be expressed as a fraction in the form of a ratio of integers. In the last lesson, we learned how to convert a repeating decimal to a fraction. Do you think that same process can be used with a number like $\sqrt{28} = 5.291502622\ldots$?
  - *No, because the decimal expansion does not seem to repeat.*

Scaffolding:

A graphic organizer may be useful. Consider the one below.

Rational Numbers	Irrational Numbers
Definition: Examples:	Definition: Examples:

- Because the number $\sqrt{28}$ cannot be expressed as a rational number, we say that it is *irrational*. Any number that cannot be expressed as a rational number is, by definition, an irrational number.
- A common irrational number is pi: $\pi = 3.14159265\ldots$. Notice that the decimal expansion of pi is infinite and does not repeat. Those qualities are what make pi an irrational number. Often for computations, we give pi a rational approximation of 3.14 or $\frac{22}{7}$, but those are merely approximations, not the true value of the number pi.
- Another example of an irrational number is $\sqrt{7}$. What do you expect the decimal expansion of $\sqrt{7}$ to look like?
  - *The decimal expansion of $\sqrt{7}$ will be infinite without a repeating block.*
- The number $\sqrt{7} = 2.645751311\ldots$. The decimal expansion is infinite and does not repeat.
- Is the number $\sqrt{49}$ rational or irrational? Explain.
  - *The number $\sqrt{49} = 7$. The decimal expansion of $\sqrt{49}$ can be written as $7.0000\ldots$, which is an infinite decimal expansion with a repeat block. Therefore, $\sqrt{49}$ is a rational number.*
- Classify the following numbers as rational or irrational. Be prepared to explain your reasoning.

$$\sqrt{10},\ 0.123123123\ldots,\ \sqrt{64},\text{ and }\frac{5}{11}$$

Provide students time to classify the numbers. They can do this independently or in pairs. Then, select students to share their reasoning. Students should identify $\sqrt{10}$ as irrational because it has a decimal expansion that can only be approximated by rational numbers. The number $0.123123123\ldots$ is a repeating decimal, can be expressed as a fraction, and is therefore rational. The number $\sqrt{64} = 8$ and is therefore a rational number. The fraction $\frac{5}{11}$ by definition is a rational number because it is a ratio of integers.

Consider going back to the Opening Exercise to determine whose approximation was the closest.

Exercise 1 (5 minutes)

Students work in pairs to complete Exercise 1.

Exercise 2

Between which interval of hundredths would $\sqrt{14}$ be located? Show your work.

The number $\sqrt{14}$ is between integers 3 and 4 because $3^2 < \left(\sqrt{14}\right)^2 < 4^2$. Then, $\sqrt{14}$ must be checked for the interval of tenths between 3 and 4. Since $\sqrt{14}$ is closer to 4, we will begin with the interval from 3.9 to 4.0. The number $\sqrt{14}$ is between 3.7 and 3.8 because $3.7^2 = 13.69$ and $3.8^2 = 14.44$. Now, we must look at the interval of hundredths between 3.7 and 3.8. Since $\sqrt{14}$ is closer to 3.7, we will begin with the interval 3.70 to 3.71. The number $\sqrt{14}$ is between 3.74 and 3.75 because $3.74^2 = 13.9876$ and $3.75^2 = 14.0625$.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that any number that cannot be expressed as a rational number is an irrational number.
- We know that to determine the approximate value of an irrational number we must determine between which two rational numbers it would lie.
- We know that the method of rational approximation uses a sequence of rational numbers, in increments of 10^0, 10^{-1}, 10^{-2}, and so on, to get closer and closer to a given number.
- We have a method for determining the approximate decimal expansion of the square root of an imperfect square, which is an irrational number.

Lesson Summary

To get the decimal expansion of a square root of a non-perfect square, you must use the method of rational approximation. Rational approximation is a method that uses a sequence of rational numbers to get closer and closer to a given number to estimate the value of the number. The method requires that you investigate the size of the number by examining its value for increasingly smaller powers of 10 (i.e., tenths, hundredths, thousandths, and so on). Since $\sqrt{22}$ is not a perfect square, you would use rational approximation to determine its decimal expansion.

Example:

Begin by determining between which two integers the number would lie.
$\sqrt{22}$ is between the integers 4 and 5 because $4^2 < \left(\sqrt{22}\right)^2 < 5^2$, which is equal to $16 < 22 < 25$.

Next, determine between which interval of tenths the number belongs.
$\sqrt{22}$ is between 4.6 and 4.7 because $4.6^2 < \left(\sqrt{22}\right)^2 < 4.7^2$, which is equal to $21.16 < 22 < 22.09$.

Next, determine between which interval of hundredths the number belongs.
$\sqrt{22}$ is between 4.69 and 4.70 because $4.69^2 < \left(\sqrt{22}\right)^2 < 4.70^2$, which is equal to $21.9961 < 22 < 22.09$.

A good estimate of the value of $\sqrt{22}$ is 4.69 because 22 is closer to 21.9961 than it is to 22.09.

Notice that with each step we are getting closer and closer to the actual value, 22. This process can continue using intervals of thousandths, ten-thousandths, and so on.

Any number that cannot be expressed as a rational number is called an irrational number. Irrational numbers are those numbers with decimal expansions that are infinite and do not have a repeating block of digits.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 11: The Decimal Expansion of Some Irrational Numbers

Exit Ticket

1. Determine the three-decimal digit approximation of the number $\sqrt{17}$.

2. Classify the following numbers as rational or irrational, and explain how you know.

$$\frac{3}{5},\ 0.73737373\ldots,\ \text{and}\ \sqrt{31}$$

Exit Ticket Sample Solutions

1. **Determine the three-decimal digit approximation of the number $\sqrt{17}$.**

   The number $\sqrt{17}$ is between integers 4 and 5 because $4^2 < \left(\sqrt{17}\right)^2 < 5^2$. Since $\sqrt{17}$ is closer to 4, I will start checking the tenths intervals closer to 4. $\sqrt{17}$ is between 4.1 and 4.2 since $4.1^2 = 16.81$ and $4.2^2 = 17.64$. Checking the hundredths interval, $\sqrt{17}$ is between 4.12 and 4.13 since $4.12^2 = 16.9744$ and $4.13^2 = 17.0569$. Checking the thousandths interval, $\sqrt{17}$ is between 4.123 and 4.124 since $4.123^2 = 16.999129$ and $4.124^2 = 17.007376$. Since 17 is closer to 4.123^2 than 4.124^2, then the three-decimal digit approximation is approximately 4.123.

2. **Classify the following numbers as rational or irrational, and explain how you know.**

   $$\frac{3}{5},\ 0.73737373\ldots,\ \text{and}\ \sqrt{31}$$

   The number $\frac{3}{5}$, by definition, is rational because it is a ratio of integers. The number $0.73737373\ldots$ is rational because it has a repeat block. For that reason, the number can be expressed as a fraction. The number $\sqrt{31}$ is irrational because it has a decimal expansion that can only be approximated by rational numbers. That is, the number is not equal to a rational number; therefore, it is irrational.

Problem Set Sample Solutions

1. **Use the method of rational approximation to determine the decimal expansion of $\sqrt{84}$. Determine which interval of hundredths it would lie in.**

   The number $\sqrt{84}$ is between 9 and 10 but closer to 9. Looking at the interval of tenths, beginning with 9.0 to 9.1, the number $\sqrt{84}$ lies between 9.1 and 9.2 because $9.1^2 = 82.81$ and $9.2^2 = 84.64$ but is closer to 9.2. In the interval of hundredths, the number $\sqrt{84}$ lies between 9.16 and 9.17 because $9.16^2 = 83.9056$ and $9.17^2 = 84.0889$.

2. **Determine the three-decimal digit approximation of the number $\sqrt{34}$.**

   The number $\sqrt{34}$ is between 5 and 6 but closer to 6. Looking at the interval of tenths, beginning with 5.9 to 6.0, the number $\sqrt{34}$ lies between 5.8 and 5.9 because $5.8^2 = 33.64$ and $5.9^2 = 34.81$ and is closer to 5.8. In the interval of hundredths, the number $\sqrt{34}$ lies between 5.83 and 5.84 because $5.83^2 = 33.9889$ and $5.84^2 = 34.1056$ and is closer to 5.83. In the interval of thousandths, the number $\sqrt{34}$ lies between 5.830 and 5.831 because $5.830^2 = 33.9889$ and $5.831^2 = 34.000561$ but is closer to 5.831. Since 34 is closer to 5.831^2 than 5.830^2, then the three-decimal digit approximation of the number is approximately 5.831.

3. **Write the decimal expansion of $\sqrt{47}$ to at least two-decimal digits.**

   The number $\sqrt{47}$ is between 6 and 7 but closer to 7 because $6^2 < \left(\sqrt{47}\right)^2 < 7^2$. In the interval of tenths, the number $\sqrt{47}$ is between 6.8 and 6.9 because $6.8^2 = 46.24$ and $6.9^2 = 47.61$. In the interval of hundredths, the number $\sqrt{47}$ is between 6.85 and 6.86 because $6.85^2 = 46.9225$ and $6.86^2 = 47.0596$. Therefore, to two-decimal digits, the number $\sqrt{47}$ is approximately 6.85 but when rounded, will be approximately 6.86 because $\sqrt{47}$ is closer to 6.86 but not quite 6.86.

4. **Write the decimal expansion of $\sqrt{46}$ to at least two-decimal digits.**

   The number $\sqrt{46}$ is between integers 6 and 7 because $6^2 < \left(\sqrt{46}\right)^2 < 7^2$. Since $\sqrt{46}$ is closer to 7, I will start checking the tenths intervals between 6.9 and 7. $\sqrt{46}$ is between 6.7 and 6.8 since $6.7^2 = 44.89$ and $6.8^2 = 46.24$. Checking the hundredths interval, $\sqrt{46}$ is between 6.78 and 6.79 since $6.78^2 = 45.9684$ and $6.79^2 = 46.1041$. Since 46 is closer to 6.78^2 than 6.79^2, then the two-decimal approximation is 6.78.

5. **Explain how to improve the accuracy of the decimal expansion of an irrational number.**

   In order to improve the accuracy of the decimal expansion of an irrational number, you must examine increasingly smaller increments on the number line. Specifically, examine increments of decreasing powers of 10. The basic inequality allows us to determine which interval a number is between. We begin by determining which two integers the number lies between and then decreasing the power of 10 to look at the interval of tenths. Again using the basic inequality, we can narrow down the approximation to a specific interval of tenths. Then we look at the interval of hundredths, and use the basic inequality to determine which interval of hundredths the number would lie between. Then we examine the interval of thousandths. Again, the basic inequality allows us to narrow down the approximation to thousandths. The more intervals we examine, the more accurate the decimal expansion of an irrational number will be.

6. **Is the number $\sqrt{125}$ rational or irrational? Explain.**

   The number $\sqrt{125}$ is an irrational number because it has a decimal expansion that is infinite and does not repeat. That is, the number $\sqrt{125}$ cannot be expressed as a rational number; therefore, it is irrational.

7. **Is the number $0.646464646\ldots$ rational or irrational? Explain.**

   The number $0.646464646\ldots = \frac{64}{99}$; therefore, it is a rational number. Not only is the number $\frac{64}{99}$ a quotient of integers, but its decimal expansion is infinite with a repeating block of digits.

8. **Is the number $3.741657387\ldots$ rational or irrational? Explain.**

   The number $3.741657387\ldots$ is an irrational number because it has a decimal expansion that is infinite and does not repeat. That is, the number $3.741657387\ldots$ cannot be expressed as a rational number; therefore, it is irrational.

9. **Is the number $\sqrt{99}$ rational or irrational? Explain.**

   The number $\sqrt{99}$ is an irrational number because it has a decimal expansion that is infinite and does not repeat. That is, the number $\sqrt{99}$ cannot be expressed as a rational number; therefore, it is irrational.

10. **Challenge: Determine the two-decimal digit approximation of the number $\sqrt[3]{9}$.**

    The number $\sqrt[3]{9}$ is between integers 2 and 3 because $2^3 < \left(\sqrt[3]{9}\right)^3 < 3^3$. Since $\sqrt[3]{9}$ is closer to 2, I will start checking the tenths intervals between 2 and 3. $\sqrt[3]{9}$ is between 2 and 2.1 since $2^3 = 8$ and $2.1^3 = 9.261$. Checking the hundredths interval, $\sqrt[3]{9}$ is between 2.08 and 2.09 since $2.08^3 = 8.998912$ and $2.09^3 = 9.129329$. Since 9 is closer to 2.08^3 than 2.09^3, the two-decimal digit approximation is 2.08.

Lesson 12: Decimal Expansions of Fractions, Part 2

Student Outcomes

- Students apply the method of rational approximation to determine the decimal expansion of a fraction.
- Students relate the method of rational approximation to the long division algorithm.

Lesson Notes

In this lesson, students use the idea of intervals of tenths, hundredths, thousandths, and so on to determine the decimal expansion of rational numbers. Since there is an explicit value that can be determined, students use what they know about mixed numbers and operations with fractions to pin down specific digits as opposed to the guess and check method used with irrational numbers. The general strategy is for students to compare a fractional value, say $\frac{2}{11}$, to a known decimal digit, that is $\frac{2}{11} = 0.1 +$ "something." Students find the difference between these two values, then work to find the next decimal digit in the expansion. The process continues until students notice a pattern in their work, leading them to recognize that the decimal expansion must be that of an infinite, repeating decimal block.

This lesson includes a fluency activity that will take approximately 10 minutes to complete. The fluency activity is a personal white board exchange with problems on volume that can be found at the end of the exercises for this lesson.

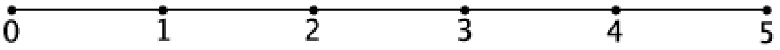
Classwork

Discussion (20 minutes)

Example 1

Write the decimal expansion of $\frac{35}{11}$.

- Our goal is to write the decimal expansion of a fraction, in this case $\frac{35}{11}$. To do so, begin by locating $\frac{35}{11}$ on the number line. What is its approximate location? Explain.



  - *The number* $\frac{35}{11}$ *would lie between* 3 *and* 4 *on the number line because* $\frac{35}{11} = \frac{33}{11} + \frac{2}{11} = 3 + \frac{2}{11}$.
- The goal is to use rational approximation to determine the decimal expansion of a number, instead of having to check a series of intervals as we did with the decimal expansions of irrational numbers. To determine the decimal expansion of $\frac{35}{11}$, focus only on the fraction $\frac{2}{11}$. Then, methodically determine between which interval of tenths $\frac{2}{11}$ would lie. Given that we are looking at an interval of tenths, can you think of a way to do this?

MP.3 Provide time for students to discuss strategies in small groups; then, share their ideas with the class. Encourage students to critique the reasoning of their classmates.

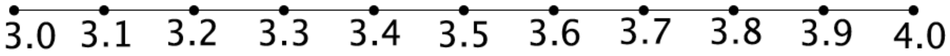
- We know that $\frac{35}{11}$ has a decimal expansion beginning with 3 in the ones place because $\frac{35}{11} = 3 + \frac{2}{11}$. Now we want to determine the tenths digit, the hundredths digit, and then the thousandths digit.

3.

Ones	Tenths	Hundredths	Thousandths

- To figure out the tenths digit, we will use an inequality based on tenths. We are looking for the consecutive integers, m and $m + 1$, that $\frac{2}{11}$ would lie between when m and $m + 1$ are intervals of tenths; in other words,

$$\frac{m}{10} < \frac{2}{11} < \frac{m+1}{10}.$$



Scaffolding:

An alternative way of asking this question is "In which interval could we place the fraction $\frac{2}{11}$?" Show students the number line labeled with tenths.

Give students time to make sense of the above inequality. Since the intervals of tenths are represented by $\frac{m}{10}$ and $\frac{m+1}{10}$, consider using concrete numbers, which is clearer than looking at consecutive intervals of tenths on the number line. The chart below may help students make sense of the intervals and the inequality.

Integer	Next Integer
1	2
3	4
5	6
12	13
114	115
m	$m+1$

Tenth	Next Tenth
$0.1 = \frac{1}{10}$	$0.2 = \frac{2}{10}$
$0.3 = \frac{3}{10}$	$0.4 = \frac{4}{10}$
$0.5 = \frac{5}{10}$	$0.6 = \frac{6}{10}$
$1.2 = \frac{12}{10}$	$1.3 = \frac{13}{10}$
$11.4 = \frac{114}{10}$	$11.5 = \frac{115}{10}$
$\frac{m}{10}$	$\frac{m+1}{10}$

- Multiplying through by 10, we get

$$m < 10\left(\frac{2}{11}\right) < m + 1$$

$$\begin{aligned} 10\left(\frac{2}{11}\right) &= \frac{20}{11} \\ &= \frac{11}{11} + \frac{9}{11} \\ &= 1 + \frac{9}{11}. \end{aligned}$$

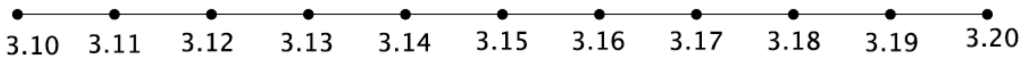
- This implies that $m = 1$. Why does the statement that $10\left(\frac{2}{11}\right) = 1 + \frac{9}{11}$ imply that $m = 1$?
    - *It implies that* $m = 1$ *because* m *and* $m + 1$ *are consecutive integers. Since* $10\left(\frac{2}{11}\right) = 1 + \frac{9}{11} = 1\frac{9}{11}$, *the number* $1\frac{9}{11}$ *would be between the two consecutive integers* 1 *and* 2, *thus implying that* $m = 1$.
- Now we know that the decimal expansion of $\frac{35}{11}$ has a one in the tenths place:

3.	1		
Ones	**Tenths**	**Hundredths**	**Thousandths**

- Since $\frac{35}{11} = 3 + \frac{2}{11}$ and the decimal expansion of the number is $3.1 = 3 + \frac{1}{10}$, we need to find the difference between these two representations. In other words, we need to find out what is left over after we remove the $\frac{1}{10}$ from the fraction $\frac{2}{11}$:

$$\frac{2}{11} - \frac{1}{10} = \frac{20}{110} - \frac{11}{110} = \frac{9}{110}.$$

- The next step is to find out which interval of hundredths will contain the fraction $\frac{9}{110}$.



Provide time for students to make a prediction and possibly develop a plan for determining the answer.

- The process is the same as looking for the interval of tenths. That is, we are looking for consecutive integers m and $m + 1$ so that

$$\frac{m}{100} < \frac{9}{110} < \frac{m+1}{100}.$$

By what number should we multiply each term of the inequality to make our work here easier?

- *Multiplying through by* 100 *will eliminate the fractions at the beginning and at the end of the inequality.*

- Multiplying through by 100, we get

$$m < \frac{900}{110} < m + 1.$$

- Between which two integers, m and $m + 1$, will we find the fraction $\frac{900}{110}$? Explain.
    - *The fraction* $\frac{900}{110}$ *is between* 8 *and* 9. *The reason is that* $\frac{900}{110} = \frac{880}{110} + \frac{20}{110} = 8 + \frac{2}{11}$.
- Now we know that the decimal expansion of $\frac{35}{11}$ has an 8 in the hundredths place:

3.	1	8	
Ones	**Tenths**	**Hundredths**	**Thousandths**

- Back to our original goal:

$$\frac{35}{11} = 3 + \frac{2}{11}.$$

By substitution we get

$$\frac{35}{11} = 3 + \frac{1}{10} + \frac{900}{110}$$
$$= 3.1 + \frac{900}{110}.$$

We know that $\frac{35}{11} = 3 + \frac{2}{11}$, and our work so far has shown the decimal expansion to be $3.18 = 3 + \frac{1}{10} + \frac{8}{100}$.

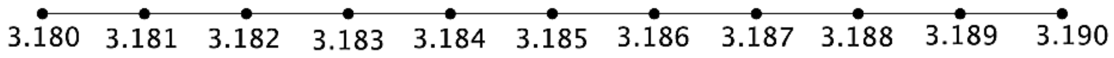
As before, we need to find the difference between $\frac{2}{11}$ and $\left(\frac{1}{10} + \frac{8}{100}\right)$:

$$\frac{2}{11} - \left(\frac{1}{10} + \frac{8}{100}\right) = \frac{2}{11} - \frac{18}{100}$$
$$= \frac{200}{1100} - \frac{198}{1100}$$
$$= \frac{2}{1100}.$$

Then, again, by substitution we get

$$\frac{35}{11} = 3 + \frac{1}{10} + \frac{900}{110}$$
$$\frac{35}{11} = 3 + \frac{1}{10} + \frac{8}{100} + \frac{2}{1100}$$
$$= 3.18 + \frac{2}{1100}.$$

- Now, look at the interval of thousandths. Where do you expect $\frac{2}{1100}$ to lie on the number line? Write and explain a plan for determining the interval of thousandths between which the number belongs.



Provide students time to make a prediction and develop a plan for determining the answer. Students should recognize that $\frac{2}{1100} = \frac{2}{11} \times \frac{1}{100}$ and that we have placed the fraction $\frac{2}{11}$ first, but for a different place value.

- Note that $\frac{2}{1100} = \frac{2}{11} \times \frac{1}{100}$. The reappearance of the fraction $\frac{2}{11}$ is meaningful in that we can expect a decimal digit to repeat, but in a different place value since we are now looking for the thousandths digit. We are looking for consecutive integers m and $m + 1$ so that

$$\frac{m}{1000} < \frac{2}{1100} < \frac{m+1}{1000}.$$

What should we multiply each term by?

  - *Multiplying through by* 1000 *will eliminate the fractions at the beginning and at the end of the inequality.*

- Multiplying through by 1000, we get
$$m < \frac{20}{11} < m + 1.$$
However, we already know that
$$\begin{aligned}\frac{20}{11} &= \frac{11}{11} + \frac{9}{11} \\ &= 1 + \frac{9}{11}.\end{aligned}$$
- Therefore, the next digit in the decimal expansion of $\frac{35}{11}$ will be 1:

3.	**1**	**8**	**1**
Ones	**Tenths**	**Hundredths**	**Thousandths**

- As before, we have the reappearance of the fraction $\frac{9}{11}$. So, we can expect the next decimal digit to be 8, followed by the reappearance of $\frac{2}{11}$, and so on. Therefore, the decimal expansion of $\frac{35}{11} = 3.1818 \ldots$.
- Perform the long division algorithm on the fraction $\frac{35}{11}$, and be prepared to share your observations.

Provide time for students to work. Ask students: How is this method of rational approximation similar to the long division algorithm? Students should notice that the algorithm became repetitive with the appearance of the numbers 2 and 9, alternating with each step. Conclude the discussion by pointing out that the method of rational approximation is similar to the long division algorithm.

Exercises 1–3 (5 minutes)

Students work independently or in pairs to complete Exercises 1–3.

Exercises 1–3

1. Use rational approximation to determine the decimal expansion of $\frac{5}{3}$.

$$\begin{aligned}\frac{5}{3} &= \frac{3}{3}+\frac{2}{3}\\ &= 1+\frac{2}{3}\end{aligned}$$

In the sequence of tenths, we are looking for integers m and $m+1$ so that

$$\frac{m}{10}<\frac{2}{3}<\frac{m+1}{10},$$

which is the same as

$$m<10\left(\frac{2}{3}\right)<m+1$$

$$\begin{aligned}\frac{20}{3} &= \frac{18}{3}+\frac{2}{3}\\ &= 6+\frac{2}{3}.\end{aligned}$$

The tenths digit is 6. The difference between $\frac{2}{3}$ and $\frac{6}{10}$ is

$$\frac{2}{3}-\frac{6}{10}=\frac{2}{30}.$$

In the interval of hundredths, we are looking for integers m and $m+1$ so that

$$\frac{m}{100}<\frac{2}{30}<\frac{m+1}{100},$$

which is the same as

$$m<\frac{20}{3}<m+1.$$

But we already know that $\frac{20}{3}=6+\frac{2}{3}$; therefore, the hundredths digit is 6. Because we keep getting $\frac{2}{3}$, we can assume the digit of 6 will continue to repeat. Therefore, the decimal expansion of $\frac{5}{3}=1.666\ldots$.

2. Use rational approximation to determine the decimal expansion of $\frac{5}{11}$.

In the sequence of tenths, we are looking for integers m *and* $m+1$ *so that*

$$\frac{m}{10} < \frac{5}{11} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{50}{11} < m+1$$

$$\frac{50}{11} = \frac{44}{11} + \frac{6}{11}$$
$$= 4 + \frac{6}{11}.$$

The tenths digit is 4. *The difference between* $\frac{5}{11}$ *and* $\frac{4}{10}$ *is*

$$\frac{5}{11} - \frac{4}{10} = \frac{6}{110}.$$

In the sequence of hundredths, we are looking for integers m *and* $m+1$ *so that*

$$\frac{m}{100} < \frac{6}{110} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{60}{11} < m+1$$

$$\frac{60}{11} = \frac{55}{11} + \frac{5}{11}$$
$$= 5 + \frac{5}{11}.$$

So, the hundredths digit is 5. *Again, we see the fraction* $\frac{5}{11}$, *which means the next decimal digit will be* 4, *as it was in the tenths place. This means we will again see the fraction* $\frac{6}{11}$, *meaning we will have another digit of* 5. *Therefore, the decimal expansion of* $\frac{5}{11}$ *is* $0.4545\ldots$.

3.

a. **Determine the decimal expansion of the number $\frac{23}{99}$ using rational approximation and long division.**

In the interval of tenths, we are looking for integers m and $m+1$ so that

$$\frac{m}{10} < \frac{23}{99} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{230}{99} < m+1$$

$$\frac{230}{99} = \frac{198}{99} + \frac{32}{99}$$
$$= 2 + \frac{32}{99}.$$

The tenths digit is 2. The difference between $\frac{23}{99}$ and $\frac{2}{10}$ is

$$\frac{23}{99} - \frac{2}{10} = \frac{32}{990}.$$

In the interval of hundredths, we are looking for integers m and $m+1$ so that

$$\frac{m}{100} < \frac{32}{99}\left(\frac{1}{10}\right) < \frac{m+1}{100},$$

which is the same as

$$m < \frac{320}{99} < m+1$$

$$\frac{320}{99} = \frac{297}{99} + \frac{23}{99}$$
$$= 3 + \frac{23}{99}.$$

So, the hundredths digit is 3. Again, we see the fraction $\frac{23}{99}$, which means the next decimal digit will be 2, as it was in the tenths place. This means we will again see the fraction $\frac{32}{99}$, meaning we will have another digit of 3. Therefore, the decimal expansion of $\frac{23}{99}$ is $0.2323\ldots$.

Long division gives us the same decimal expansion of $\frac{23}{99} = 0.2323\ldots$.

b. **When comparing rational approximation to long division, what do you notice?**

The first thing I notice is that the method of rational approximation gives the same decimal expansion as the long division algorithm. This makes sense because when doing long division, I put zeros past the 23, dividing into tenths, hundredths, thousandths, and so on. When I use the method of rational approximation, I do the same thing.

Fluency Exercise (10 minutes): Area and Volume I

RWBE: Refer to the Rapid White Board Exchanges section in the Module Overview for directions to administer a Rapid White Board Exchange.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- Using rational approximation to write the decimal expansion of a fraction is similar to using the long division algorithm.
- Use the method of rational approximation to write the decimal expansion of a fraction instead of guessing and checking the intervals of tenths, hundredths, thousandths, etc. Using computation, we can determine the interval that a decimal lies in.

Lesson Summary

The method of rational approximation, used earlier to write the decimal expansion of irrational numbers, can also be used to write the decimal expansion of fractions (rational numbers).

When used with rational numbers, there is no need to guess and check to determine the interval of tenths, hundredths, thousandths, etc., in which a number will lie. Rather, computation can be used to determine between which two consecutive integers, m and $m+1$, a number would lie for a given place value. For example, to determine where the fraction $\frac{1}{8}$ lies in the interval of tenths, compute using the following inequality:

$$\frac{m}{10} < \frac{1}{8} < \frac{m+1}{10}$$ **Use the denominator of 10 because we need to find the tenths digit of $\frac{1}{8}$.**

$$m < \frac{10}{8} < m+1$$ **Multiply through by 10.**

$$m < 1\frac{1}{4} < m+1$$ **Simplify the fraction $\frac{10}{8}$.**

The last inequality implies that $m = 1$ and $m+1 = 2$, because $1 < 1\frac{1}{4} < 2$. Then, the tenths digit of the decimal expansion of $\frac{1}{8}$ is 1.

Next, find the difference between the number $\frac{1}{8}$ and the known tenths digit value, $\frac{1}{10}$; in other words, $\frac{1}{8} - \frac{1}{10} = \frac{2}{80} = \frac{1}{40}$.

Use the inequality again, this time with $\frac{1}{40}$, to determine the hundredths digit of the decimal expansion of $\frac{1}{8}$.

$$\frac{m}{100} < \frac{1}{40} < \frac{m+1}{100}$$ **Use the denominator of 100 because of our need to find the hundredths digit of $\frac{1}{8}$.**

$$m < \frac{100}{40} < m+1$$ **Multiply through by 100.**

$$m < 2\frac{1}{2} < m+1$$ **Simplify the fraction $\frac{100}{40}$.**

The last inequality implies that $m = 2$ and $m+1 = 3$, because $2 < 2\frac{1}{2} < 3$. Then, the hundredths digit of the decimal expansion of $\frac{1}{8}$ is 2.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 12: Decimal Expansions of Fractions, Part 2

Exit Ticket

Use rational approximation to determine the decimal expansion of $\frac{41}{6}$.

Exit Ticket Sample Solutions

Use rational approximation to determine the decimal expansion of $\frac{41}{6}$.

$$\begin{aligned}\frac{41}{6} &= \frac{36}{36} + \frac{5}{6}\\ &= 6 + \frac{5}{6}\end{aligned}$$

The ones digit is 6. In the interval of tenths, we are looking for integers m and $m+1$ so that

$$\frac{m}{10} < \frac{5}{6} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{50}{6} < m+1$$

$$\begin{aligned}\frac{50}{6} &= \frac{48}{6} + \frac{2}{6}\\ &= 8 + \frac{1}{3}.\end{aligned}$$

The tenths digit is 8. The difference between $\frac{5}{6}$ and $\frac{8}{10}$ is

$$\frac{5}{6} - \frac{8}{10} = \frac{1}{30}.$$

In the interval of hundredths, we are looking for integers m and $m+1$ so that

$$\frac{m}{100} < \frac{1}{30} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{10}{3} < m+1$$

$$\begin{aligned}\frac{10}{3} &= \frac{9}{3} + \frac{1}{3}\\ &= 3 + \frac{1}{3}.\end{aligned}$$

The hundredths digit is 3. Again, we see the fraction $\frac{1}{3}$, which means the next decimal digit will be 3, as it was in the hundredths place. This means we will again see the fraction $\frac{1}{3}$, meaning we will have another digit of 3. Therefore, the decimal expansion of $\frac{41}{6}$ is $6.8333\ldots$.

Problem Set Sample Solutions

1. **Explain why the tenths digit of $\frac{3}{11}$ is 2, using rational approximation.**

In the interval of tenths, we are looking for integers m *and* $m+1$ *so that*

$$\frac{m}{10} < \frac{3}{11} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{30}{11} < m+1$$

$$\frac{30}{11} = \frac{22}{11} + \frac{8}{11}$$
$$= 2 + \frac{8}{11}.$$

In looking at the interval of tenths, we see that the number $\frac{3}{11}$ *must be between* $\frac{2}{10}$ *and* $\frac{3}{10}$ *because* $\frac{2}{10} < \frac{3}{11} < \frac{3}{10}$. *For this reason, the tenths digit of the decimal expansion of* $\frac{3}{11}$ *must be* 2.

2. **Use rational approximation to determine the decimal expansion of $\frac{25}{9}$.**

$$\frac{25}{9} = \frac{18}{9} + \frac{7}{9}$$
$$= 2 + \frac{7}{9}$$

The ones digit is 2. *In the interval of tenths, we are looking for integers* m *and* $m+1$ *so that*

$$\frac{m}{10} < \frac{7}{9} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{70}{9} < m+1$$

$$\frac{70}{9} = \frac{63}{9} + \frac{7}{9}$$
$$= 7 + \frac{7}{9}.$$

The tenths digit is 7. *The difference between* $\frac{7}{9}$ *and* $\frac{7}{10}$ *is*

$$\frac{7}{9} - \frac{7}{10} = \frac{7}{90}.$$

In the interval of hundredths, we are looking for integers m *and* $m+1$ *so that*

$$\frac{m}{100} < \frac{7}{90} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{70}{9} < m+1.$$

However, we already know that $\frac{70}{9} = 7 + \frac{7}{9}$; *therefore, the hundredths digit is* 7. *Because we keep getting* $\frac{7}{9}$, *we can assume the digit of* 7 *will continue to repeat. Therefore, the decimal expansion of* $\frac{25}{9} = 2.777\ldots$.

3. **Use rational approximation to determine the decimal expansion of $\frac{11}{41}$ to at least 5 digits.**

In the interval of tenths, we are looking for integers m and $m+1$ so that

$$\frac{m}{10} < \frac{11}{41} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{110}{41} < m+1$$

$$\frac{110}{41} = \frac{82}{41} + \frac{28}{41} = 2 + \frac{28}{41}.$$

The tenths digit is 2. The difference between $\frac{11}{41}$ and $\frac{2}{10}$ is

$$\frac{11}{41} - \frac{2}{10} = \frac{28}{410}.$$

In the interval of hundredths, we are looking for integers m and $m+1$ so that

$$\frac{m}{100} < \frac{28}{410} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{280}{41} < m+1$$

$$\frac{280}{41} = \frac{246}{41} + \frac{34}{41} = 6 + \frac{34}{41}.$$

The hundredths digit is 6. The difference between $\frac{11}{41}$ and $\left(\frac{2}{10} + \frac{6}{100}\right)$ is

$$\frac{11}{41} - \left(\frac{2}{10} + \frac{6}{100}\right) = \frac{11}{41} - \frac{26}{100} = \frac{34}{4100}.$$

In the interval of thousandths, we are looking for integers m and $m+1$ so that

$$\frac{m}{1000} < \frac{34}{4100} < \frac{m+1}{1000},$$

which is the same as

$$m < \frac{340}{41} < m+1$$

$$\frac{340}{41} = \frac{328}{41} + \frac{12}{41} = 8 + \frac{12}{41}.$$

The thousandths digit is 8. The difference between $\frac{11}{41}$ and $\left(\frac{2}{10} + \frac{6}{100} + \frac{8}{1000}\right)$ is

$$\frac{11}{41} - \left(\frac{2}{10} + \frac{6}{100} + \frac{8}{1000}\right) = \frac{11}{41} - \frac{268}{1000} = \frac{12}{41000}.$$

In the interval of ten-thousandths, we are looking for integers m and $m+1$ so that

$$\frac{m}{10000} < \frac{12}{41000} < \frac{m+1}{10000},$$

which is the same as

$$m < \frac{120}{41} < m+1$$

$$\frac{120}{41} = \frac{82}{41} + \frac{38}{41} = 2 + \frac{38}{41}.$$

The ten-thousandths digit is 2*. The difference between* $\frac{11}{41}$ *and* $\left(\frac{2}{10}+\frac{6}{100}+\frac{8}{1000}+\frac{2}{10000}\right)$ *is*

$$\frac{11}{41}-\left(\frac{2}{10}+\frac{6}{100}+\frac{8}{1000}+\frac{2}{10000}\right)=\frac{11}{41}-\frac{2682}{10000}=\frac{38}{410000}.$$

In the interval of hundred-thousandths, we are looking for integers m *and* $m+1$ *so that*

$$\frac{m}{100000}<\frac{38}{410000}<\frac{m+1}{100000},$$

which is the same as

$$m<\frac{380}{41}<m+1$$

$$\frac{380}{41}=\frac{369}{41}+\frac{11}{41}=9+\frac{11}{41}.$$

The hundred-thousandths digit is 9*. We see again the fraction* $\frac{11}{41}$*, so we can expect the decimal digits to repeat at this point. Therefore, the decimal approximation of* $\frac{11}{41}=0.2682926829\ldots$*.*

4. Use rational approximation to determine which number is larger, $\sqrt{10}$ or $\frac{28}{9}$.

The number $\sqrt{10}$ *is between* 3 *and* 4*. In the sequence of tenths,* $\sqrt{10}$ *is between* 3.1 *and* 3.2 *because* $3.1^2<\left(\sqrt{10}\right)^2<3.2^2$*. In the sequence of hundredths,* $\sqrt{10}$ *is between* 3.16 *and* 3.17 *because* $3.16^2<\left(\sqrt{10}\right)^2<3.17^2$*. In the sequence of hundredths,* $\sqrt{10}$ *is between* 3.162 *and* 3.163 *because* $3.162^2<\left(\sqrt{10}\right)^2<3.163^2$*. The decimal expansion of* $\sqrt{10}$ *is approximately* $3.162\ldots$*.*

$$\begin{aligned}\frac{28}{9}&=\frac{27}{9}+\frac{1}{9}\\&=3+\frac{1}{9}\end{aligned}$$

In the interval of tenths, we are looking for the integers m *and* $m+1$ *so that*

$$\frac{m}{10}<\frac{1}{9}<\frac{m+1}{10},$$

which is the same as

$$m<\frac{10}{9}<m+1$$

$$\begin{aligned}\frac{10}{9}&=\frac{9}{9}+\frac{1}{9}\\&=1+\frac{1}{9}.\end{aligned}$$

The tenths digit is 1*. Since the fraction* $\frac{1}{9}$ *has reappeared, then we can assume that the next digit is also* 1*, and the work will continue to repeat. Therefore, the decimal expansion of* $\frac{28}{9}=3.1111\ldots$*.*

Therefore, $\frac{28}{9}<\sqrt{10}$*.*

5. Sam says that $\frac{7}{11} = 0.63$, and Jaylen says that $\frac{7}{11} = 0.636$. Who is correct? Why?

In the interval of tenths, we are looking for integers m and $m+1$ so that

$$\frac{m}{10} < \frac{7}{11} < \frac{m+1}{10},$$

which is the same as

$$m < \frac{70}{11} < \frac{m+1}{10}$$

$$\frac{70}{11} = \frac{66}{11} + \frac{4}{11}$$

$$= 6 + \frac{4}{11}.$$

The tenths digit is 6. The difference between $\frac{7}{11}$ and $\frac{6}{10}$ is

$$\frac{7}{11} - \frac{6}{10} = \frac{4}{110}.$$

In the interval of hundredths, we are looking for integers m and $m+1$ so that

$$\frac{m}{100} < \frac{4}{110} < \frac{m+1}{100},$$

which is the same as

$$m < \frac{40}{11} < m+1$$

$$\frac{40}{11} = \frac{33}{11} + \frac{7}{11}$$

$$= 3 + \frac{7}{11}.$$

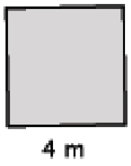
The hundredths digit is 3. Again, we see the fraction $\frac{7}{11}$, which means the next decimal digit will be 6, as it was in the tenths place. This means we will again see the fraction $\frac{4}{11}$, meaning we will have another digit of 3. Therefore, the decimal expansion of $\frac{7}{11}$ is $0.6363\ldots$.

Then, technically, both Sam and Jaylen are incorrect because the fraction $\frac{7}{11}$ is an infinite decimal. However, Sam is correct to the first two decimal digits of the number, and Jaylen is correct to the first three decimal digits of the number.

Area and Volume I

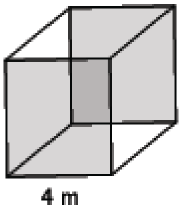
1. Find the area of the square shown below.

$$A = (4\text{ m})^2$$
$$= 16\text{ m}^2$$



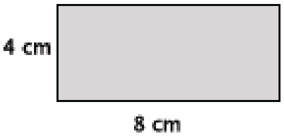
2. Find the volume of the cube shown below.

$$V = (4\text{ m})^3$$
$$= 64\text{ m}^3$$



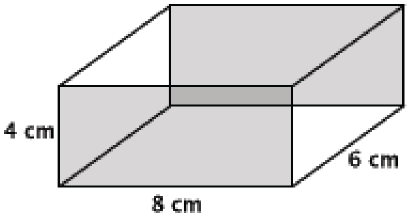
3. Find the area of the rectangle shown below.

$$A = (8\text{ cm})(4\text{ cm})$$
$$= 32\text{ cm}^2$$



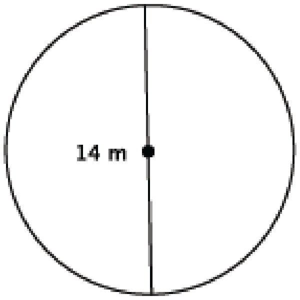
4. Find the volume of the rectangular prism shown below.

$$V = (32\text{ cm}^2)(6\text{ cm})$$
$$= 192\text{ cm}^3$$



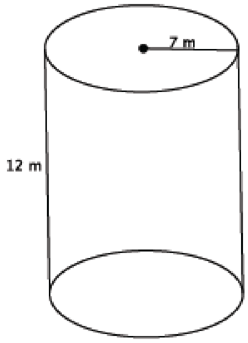
5. Find the area of the circle shown below.

$$A = (7\text{ m})^2\pi$$
$$= 49\pi\text{ m}^2$$



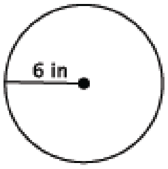
6. Find the volume of the cylinder shown below.

$$V = (49\pi\ \text{m}^2)(12\ \text{m})$$
$$= 588\pi\ \text{m}^3$$



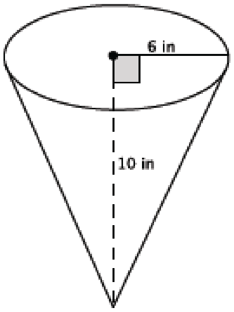
7. Find the area of the circle shown below.

$$A = (6\ \text{in.})^2\pi$$
$$= 36\pi\ \text{in}^2$$



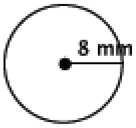
8. Find the volume of the cone shown below.

$$V = \left(\frac{1}{3}\right)(36\pi\ \text{in}^2)(10\ \text{in.})$$
$$= 120\pi\ \text{in}^3$$



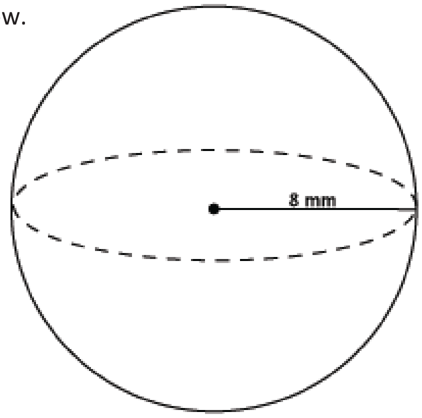
9. Find the area of the circle shown below.

$$A = (8\ \text{mm})^2\pi$$
$$= 64\pi\ \text{mm}^2$$



10. Find the volume of the sphere shown below.

$$V = \left(\frac{4}{3}\right)\pi(64\ \text{mm}^2)(8\ \text{mm})$$
$$= \frac{2048}{3}\pi\ \text{mm}^3$$



Lesson 13: Comparing Irrational Numbers

Student Outcomes

- Students use rational approximations of irrational numbers to compare the size of irrational numbers.
- Students place irrational numbers in their approximate locations on a number line.

Classwork

Exploratory Challenge/Exercises 1–11 (30 minutes)

Students work in pairs to complete Exercises 1–11. The first exercise may be used to highlight the process of answering and explaining the solution to each question. An emphasis should be placed on students' ability to explain their reasoning. Consider allowing students to use a calculator to check their work, but all decimal expansions should be done by hand. At the end of the Exploratory Challenge, consider asking students to state or write a description of their approach to solving each exercise.

MP.1 & MP.3

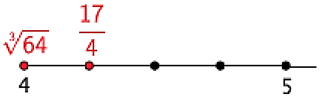
Exploratory Challenge/Exercises 1–11

1. Rodney thinks that $\sqrt[3]{64}$ is greater than $\frac{17}{4}$. Sam thinks that $\frac{17}{4}$ is greater. Who is right and why?

$\sqrt[3]{64} = \sqrt[3]{4^3} = 4$

$$\begin{aligned}\frac{17}{4} &= \frac{16}{4} + \frac{1}{4}\\ &= 4 + \frac{1}{4}\\ &= 4\frac{1}{4}\end{aligned}$$

Because $4 < 4\frac{1}{4}$, then $\sqrt[3]{64} < \frac{17}{4}$. So, $\sqrt[3]{64}$ is smaller than $\frac{17}{4}$. The number $\frac{17}{4}$ is equal to the mixed number $4\frac{1}{4}$. The cube root of 64 is the whole number 4. Because $4\frac{1}{4}$ is to the right of 4 on the number line, then $4\frac{1}{4}$ is greater than 4, which means that $\frac{17}{4} > \sqrt[3]{64}$; therefore, Sam is correct.



2. Which number is smaller, $\sqrt[3]{27}$ or 2.89? Explain.

$$\sqrt[3]{27} = \sqrt[3]{3^3} = 3$$

Because $2.89 < 3$, *then* $2.89 < \sqrt[3]{27}$; *therefore,* 2.89 *is smaller than* $\sqrt[3]{27}$. *On a number line,* 3 *is to the right of* 2.89, *meaning that* 3 *is greater than* 2.89. *Therefore,* $2.89 < \sqrt[3]{27}$.

3. Which number is smaller, $\sqrt{121}$ or $\sqrt[3]{125}$? Explain.

$$\sqrt{121} = \sqrt{11^2} = 11$$

$$\sqrt[3]{125} = \sqrt[3]{5^3} = 5$$

Because $5 < 11$, *then* $\sqrt[3]{125} < \sqrt{121}$. *So,* $\sqrt[3]{125}$ *is smaller. On a number line, the number* 5 *is to the left of* 11, *meaning that* 5 *is less than* 11. *Therefore,* $\sqrt[3]{125} < \sqrt{121}$.

4. Which number is smaller, $\sqrt{49}$ or $\sqrt[3]{216}$? Explain.

$$\sqrt{49} = \sqrt{7^2} = 7$$

$$\sqrt[3]{216} = \sqrt[3]{6^3} = 6$$

Because $6 < 7$, *then* $\sqrt[3]{216} < \sqrt{49}$. *So,* $\sqrt[3]{216}$ *is smaller than* $\sqrt{49}$. *On the number line,* 7 *is to the right of* 6, *meaning that* 7 *is greater than* 6. *Therefore,* $\sqrt[3]{216} < \sqrt{49}$.

5. Which number is greater, $\sqrt{50}$ or $\frac{319}{45}$? Explain.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{319}{45}$ *is equal to* $7.0\overline{8}$.

The number $\sqrt{50}$ *is between* 7 *and* 8 *because* $7^2 < \left(\sqrt{50}\right)^2 < 8^2$. *The number* $\sqrt{50}$ *is between* 7.0 *and* 7.1 *because* $7^2 < \left(\sqrt{50}\right)^2 < 7.1^2$. *The number* $\sqrt{50}$ *is between* 7.07 *and* 7.08 *because* $7.07^2 < \left(\sqrt{50}\right)^2 < 7.08^2$. *The approximate decimal value of* $\sqrt{50}$ *is* 7.07. *Since* $7.07 < 7.0\overline{8}$, *then* $\sqrt{50} < \frac{319}{45}$; *therefore, the fraction* $\frac{319}{45}$ *is greater than* $\sqrt{50}$.

6. Which number is greater, $\frac{5}{11}$ or $0.\overline{4}$? Explain.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{5}{11}$ *is equal to* $0.\overline{45}$. *Since* $0.44444\ldots < 0.454545\ldots$, *then* $0.\overline{4} < \frac{5}{11}$; *therefore, the fraction* $\frac{5}{11}$ *is greater than* $0.\overline{4}$.

7. **Which number is greater, $\sqrt{38}$ or $\frac{154}{25}$? Explain.**

Note that students may have used long division or equivalent fractions to determine the decimal expansion of the fraction.

$$\frac{154}{25} = \frac{154 \times 4}{25 \times 4} = \frac{616}{10^2} = 6.16$$

The number $\sqrt{38}$ *is between* 6 *and* 7 *because* $6^2 < \left(\sqrt{38}\right)^2 < 7^2$. *The number* $\sqrt{38}$ *is between* 6.1 *and* 6.2 *because* $6.1^2 < \left(\sqrt{38}\right)^2 < 6.2^2$. *The number* $\sqrt{38}$ *is between* 6.16 *and* 6.17 *because* $6.16^2 < \left(\sqrt{38}\right)^2 < 6.17^2$. *Since* $\sqrt{38}$ *is greater than* 6.16, *then* $\sqrt{38}$ *is greater than* $\frac{154}{25}$.

8. **Which number is greater, $\sqrt{2}$ or $\frac{15}{9}$? Explain.**

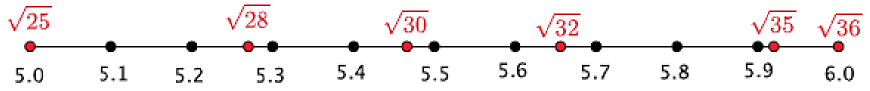
Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{15}{9}$ *is equal to* $1.\overline{6}$.

The number $\sqrt{2}$ *is between* 1 *and* 2 *because* $1^2 < \left(\sqrt{2}\right)^2 < 2^2$. *The number* $\sqrt{2}$ *is between* 1.4 *and* 1.5 *because* $1.4^2 < \left(\sqrt{2}\right)^2 < 1.5^2$. *Therefore,* $\sqrt{2} < \frac{15}{9}$; *the fraction* $\frac{15}{9}$ *is greater.*

9. **Place each of the following numbers at its approximate location on the number line: $\sqrt{25}$, $\sqrt{28}$, $\sqrt{30}$, $\sqrt{32}$, $\sqrt{35}$, and $\sqrt{36}$.**

Solutions shown in red:



The number $\sqrt{25} = \sqrt{5^2} = 5$.

The number $\sqrt{28}$ *is between* 5 *and* 6. *The number* $\sqrt{28}$ *is between* 5.2 *and* 5.3 *because* $5.2^2 < \left(\sqrt{28}\right)^2 < 5.3^2$.
The number $\sqrt{30}$ *is between* 5 *and* 6. *The number* $\sqrt{30}$ *is between* 5.4 *and* 5.5 *because* $5.4^2 < \left(\sqrt{30}\right)^2 < 5.5^2$.
The number $\sqrt{32}$ *is between* 5 *and* 6. *The number* $\sqrt{32}$ *is between* 5.6 *and* 5.7 *because* $5.6^2 < \left(\sqrt{32}\right)^2 < 5.7^2$.
The number $\sqrt{35}$ *is between* 5 *and* 6. *The number* $\sqrt{35}$ *is between* 5.9 *and* 6.0 *because* $5.9^2 < \left(\sqrt{35}\right)^2 < 6^2$.

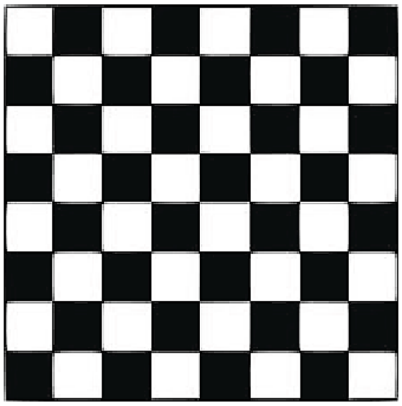
The number $\sqrt{36} = \sqrt{6^2} = 6$.

10. **Challenge: Which number is larger $\sqrt{5}$ or $\sqrt[3]{11}$?**

The number $\sqrt{5}$ *is between* 2 *and* 3 *because* $2^2 < \left(\sqrt{5}\right)^2 < 3^2$. *The number* $\sqrt{5}$ *is between* 2.2 *and* 2.3 *because* $2.2^2 < \left(\sqrt{5}\right)^2 < 2.3^2$. *The number* $\sqrt{5}$ *is between* 2.23 *and* 2.24 *because* $2.23^2 < \left(\sqrt{5}\right)^2 < 2.24^2$. *The number* $\sqrt{5}$ *is between* 2.236 *and* 2.237 *because* $2.236^2 < \left(\sqrt{5}\right)^2 < 2.237^2$. *The decimal expansion of* $\sqrt{5}$ *is approximately* $2.236 \ldots$.

The number $\sqrt[3]{11}$ *is between* 2 *and* 3 *because* $2^3 < \left(\sqrt[3]{11}\right)^3 < 3^3$. *The number* $\sqrt[3]{11}$ *is between* 2.2 *and* 2.3 *because* $2.2^3 < \left(\sqrt[3]{11}\right)^3 < 2.3^3$. *The number* $\sqrt[3]{11}$ *is between* 2.22 *and* 2.23 *because* $2.22^3 < \left(\sqrt[3]{11}\right)^3 < 2.23^3$. *The decimal expansion of* $\sqrt[3]{11}$ *is approximately* $2.22 \ldots$. *Since* $2.22 \ldots < 2.236 \ldots$, *then* $\sqrt[3]{11} < \sqrt{5}$; *therefore,* $\sqrt{5}$ *is larger.*

11. **A certain chessboard is being designed so that each square has an area of 3 in^2. What is the length of one edge of the board rounded to the tenths place? (A chessboard is composed of 64 squares as shown.)**



The area of one square is 3 in^2. So, if x is the length of one side of one square,

$$x^2 = 3$$
$$\sqrt{x^2} = \sqrt{3}$$
$$x = \sqrt{3}.$$

There are 8 squares along one edge of the board, so the length of one edge is $8 \times \sqrt{3}$. The number $\sqrt{3}$ is between 1 and 2 because $1^2 < \left(\sqrt{3}\right)^2 < 2^2$. The number $\sqrt{3}$ is between 1.7 and 1.8 because $1.7^2 < \left(\sqrt{3}\right)^2 < 1.8^2$. The number $\sqrt{3}$ is between 1.73 and 1.74 because $1.73^2 < \left(\sqrt{3}\right)^2 < 1.74^2$. The number $\sqrt{3}$ is approximately 1.73. So, the length of one edge of the chessboard is $8 \times 1.73 = 13.84\ldots$, which is approximately 13.8 in.

Note: Some students may determine the total area of the board, $64 \times 3 = 192$, then determine the approximate value of $\sqrt{192} \approx 13.8$ to answer the question.

Discussion (5 minutes)

- How do we know if a number is rational or irrational?
  - *Numbers that can be expressed as a fraction, i.e., a ratio of integers, are by definition rational numbers. Any number that is not rational is irrational.*
- Is the number $1.\overline{6}$ rational or irrational? Explain.
  - *The number $1.\overline{6}$ is rational because it is equal to $\frac{15}{9}$.*
- Is the number $\sqrt{2}$ rational or irrational? Explain.
  - *Since $\sqrt{2}$ is not a perfect square, then $\sqrt{2}$ is an irrational number. This means that the decimal expansion can only be approximated by rational numbers.*
- Which strategy do you use to write the decimal expansion of a fraction? What strategy do you use to write the decimal expansion of square and cube roots?
  - *Student responses will vary. Students will likely state that they use long division or equivalent fractions to write the decimal expansion of fractions. Students will say that they have to use the definition of square and cube roots or rational approximation to write the decimal expansion of square and cube roots.*

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- The decimal expansion of rational numbers that are expressed as fractions can be found using long division, using what we know about equivalent fractions for finite decimals, or using rational approximation.
- The approximate decimal expansions of irrational numbers (square roots of imperfect squares and imperfect cubes) can be found using rational approximation.
- Numbers, of any form (e.g., fraction, decimal, square root), can be ordered and placed in their approximate location on a number line.

Lesson Summary

The decimal expansion of rational numbers can be found using long division, equivalent fractions, or the method of rational approximation.

The decimal expansion of irrational numbers can be found using the method of rational approximation.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 13: Comparing Irrational Numbers

Exit Ticket

Place each of the following numbers at its approximate location on the number line: $\sqrt{12}$, $\sqrt{16}$, $\frac{20}{6}$, $3.\overline{53}$, and $\sqrt[3]{27}$.

Exit Ticket Sample Solutions

Place each of the following numbers at its approximate location on the number line: $\sqrt{12}$, $\sqrt{16}$, $\frac{20}{6}$, $3.\overline{53}$, and $\sqrt[3]{27}$.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

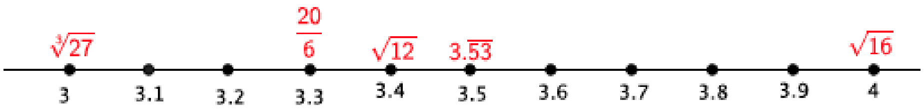
The number $\sqrt{12}$ *is between* 3.4 *and* 3.5*, since* $3.4^2 < (\sqrt{12})^2 < 3.5^2$.

The number $\sqrt{16} = \sqrt{4^2} = 4$.

The number $\frac{20}{6}$ *is equal to* $3.\overline{3}$.

The number $\sqrt[3]{27} = \sqrt[3]{3^3} = 3$.

Solutions shown in red:



Problem Set Sample Solutions

1. Which number is smaller, $\sqrt[3]{343}$ or $\sqrt{48}$? Explain.

$$\sqrt[3]{343} = \sqrt[3]{7^3} = 7$$

The number $\sqrt{48}$ *is between* 6 *and* 7*, but definitely less than* 7*. Therefore,* $\sqrt{48} < \sqrt[3]{343}$*, and* $\sqrt{48}$ *is smaller.*

2. Which number is smaller, $\sqrt{100}$ or $\sqrt[3]{1000}$? Explain.

$$\sqrt{100} = \sqrt{10^2} = 10$$
$$\sqrt[3]{1000} = \sqrt[3]{10^3} = 10$$

The numbers $\sqrt{100}$ *and* $\sqrt[3]{1000}$ *are equal because both are equal to* 10.

3. Which number is larger, $\sqrt{87}$ or $\frac{929}{99}$? Explain.

Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{929}{99}$ *is equal to* $9.\overline{38}$.

The number $\sqrt{87}$ *is between* 9 *and* 10 *because* $9^2 < \left(\sqrt{87}\right)^2 < 10^2$*. The number* $\sqrt{87}$ *is between* 9.3 *and* 9.4 *because* $9.3^2 < \left(\sqrt{87}\right)^2 < 9.4^2$*. The number* $\sqrt{87}$ *is between* 9.32 *and* 9.33 *because* $9.32^2 < \left(\sqrt{87}\right)^2 < 9.33^2$*. The approximate decimal value of* $\sqrt{87}$ *is* 9.32 *…. Since* $9.32 < 9.\overline{38}$*, then* $\sqrt{87} < \frac{929}{99}$*; therefore, the fraction* $\frac{929}{99}$ *is larger.*

4. Which number is larger, $\frac{9}{13}$ or $0.\overline{692}$? Explain.

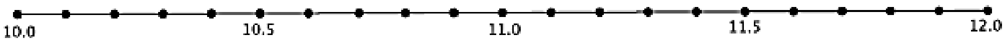
Note that students may have used long division or the method of rational approximation to determine the decimal expansion of the fraction.

The number $\frac{9}{13}$ *is equal to* $0.\overline{692307}$. *Since* $0.692307\ldots < 0.692692\ldots$, *then* $\frac{9}{13} < 0.\overline{692}$; *therefore, the decimal* $0.\overline{692}$ *is larger.*

5. Which number is larger, 9.1 or $\sqrt{82}$? Explain.

The number $\sqrt{82}$ *is between* 9 *and* 10 *because* $9^2 < \left(\sqrt{82}\right)^2 < 10^2$. *The number* $\sqrt{82}$ *is between* 9.0 *and* 9.1 *because* $9.0^2 < \left(\sqrt{82}\right)^2 < 9.1^2$. *Since* $\sqrt{82} < 9.1$, *then the number* 9.1 *is larger than the number* $\sqrt{82}$.

6. Place each of the following numbers at its approximate location on the number line: $\sqrt{144}$, $\sqrt[3]{1000}$, $\sqrt{130}$, $\sqrt{110}$, $\sqrt{120}$, $\sqrt{115}$, and $\sqrt{133}$. Explain how you knew where to place the numbers.



Solutions shown in red:

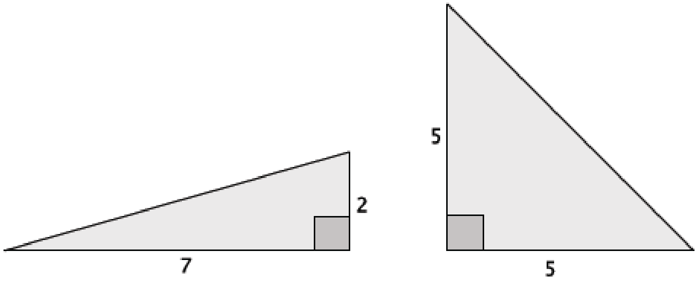
The number $\sqrt{144} = \sqrt{12^2} = 12$.

The number $\sqrt[3]{1000} = \sqrt[3]{10^3} = 10$.

The numbers $\sqrt{110}$, $\sqrt{115}$, *and* $\sqrt{120}$ *are all between* 10 *and* 11 *because when squared, their value falls between* 10^2 *and* 11^2. *The number* $\sqrt{110}$ *is between* 10.4 *and* 10.5 *because* $10.4^2 < \left(\sqrt{110}\right)^2 < 10.5^2$. *The number* $\sqrt{115}$ *is between* 10.7 *and* 10.8 *because* $10.7^2 < \left(\sqrt{115}\right)^2 < 10.8^2$. *The number* $\sqrt{120}$ *is between* 10.9 *and* 11 *because* $10.9^2 < \left(\sqrt{120}\right)^2 < 11^2$.

The numbers $\sqrt{130}$ *and* $\sqrt{133}$ *are between* 11 *and* 12 *because when squared, their value falls between* 11^2 *and* 12^2. *The number* $\sqrt{130}$ *is between* 11.4 *and* 11.5 *because* $11.4^2 < \left(\sqrt{130}\right)^2 < 11.5^2$. *The number* $\sqrt{133}$ *is between* 11.5 *and* 11.6 *because* $11.5^2 < \left(\sqrt{133}\right)^2 < 11.6^2$.

7. **Which of the two right triangles shown below, measured in units, has the longer hypotenuse? Approximately how much longer is it?**



Let x represent the hypotenuse of the triangle on the left.

$$7^2 + 2^2 = x^2$$
$$49 + 4 = x^2$$
$$53 = x^2$$
$$\sqrt{53} = \sqrt{x^2}$$
$$\sqrt{53} = x$$

The number $\sqrt{53}$ is between 7 and 8 because $7^2 < \left(\sqrt{53}\right)^2 < 8^2$. The number $\sqrt{53}$ is between 7.2 and 7.3 because $7.2^2 < \left(\sqrt{53}\right)^2 < 7.3^2$. The number $\sqrt{53}$ is between 7.28 and 7.29 because $7.28^2 < \left(\sqrt{53}\right)^2 < 7.29^2$. The approximate decimal value of $\sqrt{53}$ is 7.28 ….

Let y represent the hypotenuse of the triangle on the right.

$$5^2 + 5^2 = y^2$$
$$25 + 25 = y^2$$
$$50 = y^2$$
$$\sqrt{50} = \sqrt{y^2}$$
$$\sqrt{50} = y$$

The number $\sqrt{50}$ is between 7 and 8 because $7^2 < \left(\sqrt{50}\right)^2 < 8^2$. The number $\sqrt{50}$ is between 7.0 and 7.1 because $7.0^2 < \left(\sqrt{50}\right)^2 < 7.1^2$. The number $\sqrt{50}$ is between 7.07 and 7.08 because $7.07^2 < \left(\sqrt{50}\right)^2 < 7.08^2$. The approximate decimal value of $\sqrt{50}$ is 7.07 ….

The triangle on the left has the longer hypotenuse. It is approximately 0.21 units longer than the hypotenuse of the triangle on the right.

Note: Based on their experience, some students may reason that $\sqrt{50} < \sqrt{53}$. To answer completely, students must determine the decimal expansion to approximate how much longer one hypotenuse is than the other.

Lesson 14: Decimal Expansion of π

Student Outcomes

- Students calculate the decimal expansion of π using basic properties of area.
- Students estimate the value of numbers such as π^2.

Lesson Notes

For this lesson, students will need grid paper and a compass. Lead students through the activity that produces the decimal expansion of π. Quarter circles on grids of 10 by 10 and 20 by 20 are included at the end of the lesson if you would prefer to hand out the grids as opposed to students making their own with grid paper and a compass.

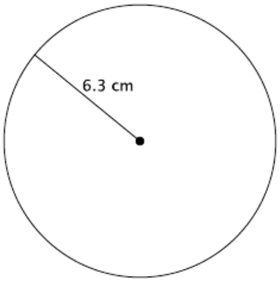
Classwork

Opening Exercise (5 minutes)

The purpose of the Opening Exercise is to activate students' prior knowledge of π and of what that number means.

Opening Exercise

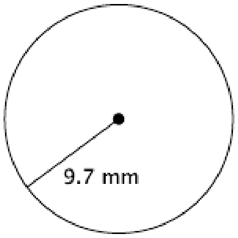
a. **Write an equation for the area, A, of the circle shown.**



$$A = \pi 6.3^2$$
$$= 39.69\pi$$

The area of the circle is $39.69\pi\ \text{cm}^2$.

b. **Write an equation for the circumference, C, of the circle shown.**



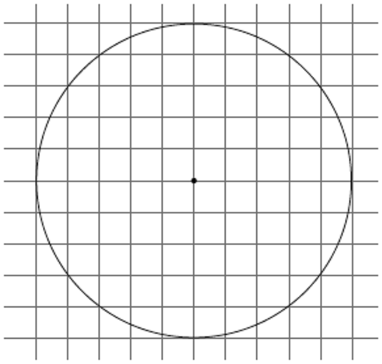
$$C = 2\pi(9.7)$$
$$= 19.4\pi$$

The circumference of the circle is 19.4π mm.

c. **Each of the squares in the grid below has an area of 1 unit^2.**

i. **Estimate the area of the circle shown by counting squares.**

Estimates will vary. The approximate area of the circle is 78 units^2.



ii. **Calculate the area of the circle using a radius of 5 units. Use 3.14 as an approximation for π.**

$$A \approx 3.14(5^2)$$
$$\approx 78.5$$

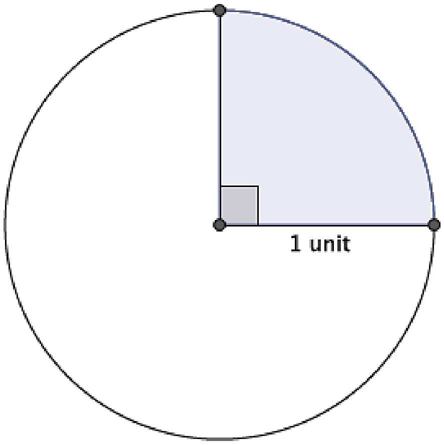
The area of the circle is approximately 78.5 units^2.

Discussion (25 minutes)

- The number pi, π, is defined as the area of a unit circle. A unit circle is a circle with a radius of one unit. In the past, you may have thought of the number π as the ratio of the circumference to the diameter of a circle. Our goal in this lesson is to determine the decimal expansion of π. What do you think that is?
  - *Students will likely state that the decimal expansion of* π *is* 3.14 *because that is the number they have used in the past to approximate* π.
- The number 3.14 is often used to approximate π, but it is not its decimal expansion. How might we determine its real decimal expansion?

Provide time for students to try to develop a plan for determining the decimal expansion of π. Have students share their ideas with the class.

- To determine the decimal expansion of π, we will use the fact that the number π is the area of a unit circle together with the counting strategy used in Opening Exercise, part (c), subpart (i). Since the area of the unit circle is equal to π, and we will be counting squares, we can decrease our work by focusing on the area of just $\frac{1}{4}$ of the circle. What is the area of $\frac{1}{4}$ of a unit circle?

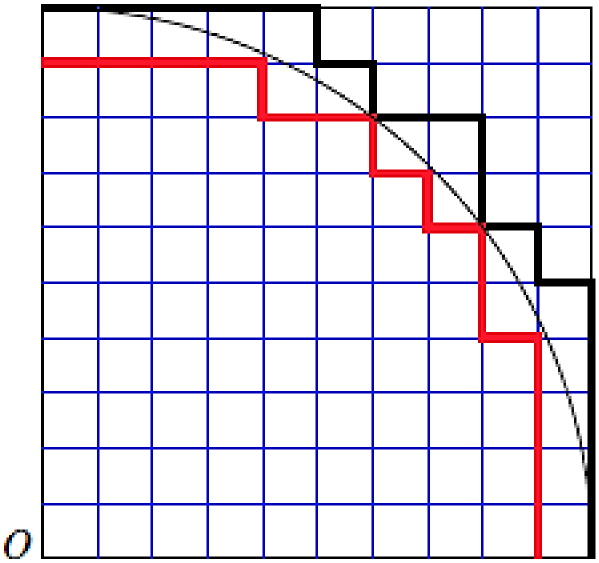


  - *Since the unit circle has an area of* π, *then* $\frac{1}{4}\pi$ *will be the area of* $\frac{1}{4}$ *of the unit circle.*

- On a piece of graph paper, mark a center O, near the center of the paper. Use your ruler and draw two lines through O, one horizontal and one vertical. Our unit will be 10 of the grid squares on the graph paper. Use your compass to measure 10 of the grid squares, and then make an arc to represent the outer edge of the quarter circle. Make sure your arc intersects the horizontal and vertical lines you drew.

Verify that all students have a quarter circle on their graph paper.

- What we have now are inner squares, which are those that are inside the quarter circle, and outer squares, which are those that are outside the quarter circle. What we want to do is mark a border just inside the circle and just outside the circle, but as close to the arc of the circle as possible. Mark a border inside the circle that captures all of the whole squares; you should not include any partial squares in this border (shown in red below). Mark a border just outside the arc that contains all of the whole squares within the quarter circle and parts of the squares that are just outside the circle (shown in black below).



- The squares of the grid paper are congruent; that is, they are all equal in size and, thus, area. We will let r_2 denote the totality of all of the inner squares and s_2 the totality of all of the outer squares. Then clearly,

$$r_2 < \frac{\pi}{4} < s_2.$$

- Count how many squares are contained within r_2 and s_2.
    - *There are* 69 *inner squares and* 86 *outer squares.*
- If we consider the area of the square with side length equal to 10 squares of the grid paper, then $r_2 = \frac{69}{100}$. What does s_2 equal?
    - *The area of* $s_2 = \frac{86}{100}$.

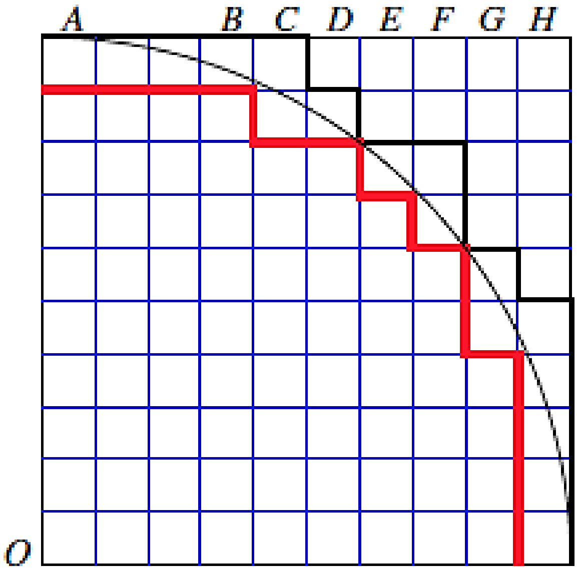
- By substitution we see that

$$r_2 < \frac{\pi}{4} < s_2$$
$$\frac{69}{100} < \frac{\pi}{4} < \frac{86}{100}$$
$$0.69 < \frac{\pi}{4} < 0.86.$$

Multiplying by 4 throughout gives

$$2.76 < \pi < 3.44.$$

- Is this inequality showing a value for π that we know to be accurate? Explain.
  - *Yes, because we frequently use* 3.14 *to represent* π*, and* $2.76 < 3.14 < 3.44$*.*
- Of course we can improve our estimate of π by taking another look at those grid squares. Columns have been labeled A–H at the top.



MP.6

- Look at the top row, columns A through B. There are some significant portions of squares that were not included in the area of the quarter circle. If we wanted to represent that portion of the circle with a whole number of grid squares, about how many do you think it would be?

Accept any reasonable answers that students give for this and the next few questions about columns A–G. Included in the text below is a possible scenario; however, your students may make better estimations and decide on different numbers of squares to include in the area.

  - *It looks like there would be at least* 2 *whole squares but likely less than* 3*.*
- Now look at columns C and D. Using similar reasoning, about how many grid squares do you think we should add to the area of the quarter circle using just columns C and D?
  - *It looks like we should add* 1 *more to the area of the quarter circle.*
- Now look at columns E and F. What should we add to the area of the quarter circle?
  - *We should add* 1 *more to the area.*

- What about column G?
    - *We should add at least* 1 *more square to the area.*
- Finally, look at column H. What should we add to the area to represent the portion of the quarter circle not accounted for yet?
    - *It looks like column H is just like columns A to B, so we should add* 2 *more to the area.*
- We began by counting only the number of whole squares within the border of the quarter circle, which totals 69. By estimating partial amounts of squares in columns A through H, we have decided to improve our estimate by adding another $2, 1, 1, 1$, and 2 squares, making our total number of grid squares represented by the quarter circle 76. Therefore, we have refined r_2 to 76, which means that

$$\frac{76}{100} < \frac{\pi}{4} < \frac{86}{100},$$

which is equal to

$$\frac{304}{100} < \pi < \frac{344}{100}$$
$$3.04 < \pi < 3.44.$$

MP.6

Does this inequality still represent a value we expect π to be?

- *Yes, because* $3.04 < 3.14 < 3.44$.

- We can reason the same way as before to refine the estimate of s_2.

Provide students time to refine their estimate of s_2. It is likely that students will come up with different numbers, but they should all be very close. Expect students to say that they have refined their estimate of s_2 to 80, instead of the original 86.

- Thus, we have

$$\frac{76}{100} < \frac{\pi}{4} < \frac{80}{100}$$
$$\frac{304}{100} < \pi < \frac{320}{100}$$
$$3.04 < \pi < 3.20.$$

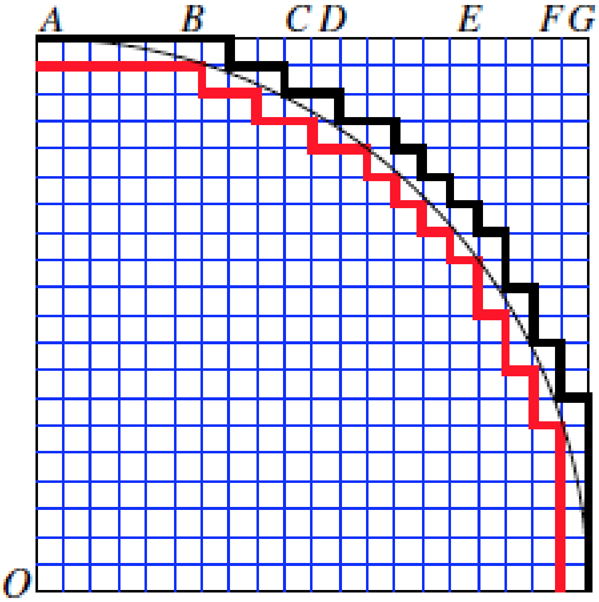
Does this inequality still represent a value we expect π to be?

- *Yes, because* $3.04 < 3.14 < 3.20$.

MP.8

- These are certainly respectable approximations of π. What would make our approximation better?
    - *We could decrease the size of the squares we are using to develop the area of the quarter circle. We could go back and make better estimations of the squares to include in* r_2 *and the squares not to include in* s_2.

- As you have stated, one way to improve our approximation is by using smaller squares. Suppose we divide each square horizontally and vertically so that instead of having 100 squares, we have 400 squares.



If time permits, allow students to repeat the process that we just went through when we had only 100 squares in the unit square. If time does not permit, then provide them with the information below.

- Then, the inner region, r_2, is composed of 294 squares, and the outer region, s_2, is composed of 333 squares. This means that

$$\frac{294}{400} < \frac{\pi}{4} < \frac{333}{400}.$$

Multiplying by 4 throughout, we have

$$\frac{294}{100} < \pi < \frac{333}{100}$$
$$2.94 < \pi < 3.33.$$

- By looking at partial squares that can be combined, the refined estimate of r_2 is 310, and s_2 is 321. Then, the inequality is

$$\frac{310}{400} < \frac{\pi}{4} < \frac{321}{400}$$
$$\frac{310}{100} < \pi < \frac{321}{100}$$
$$3.10 < \pi < 3.21.$$

- How does this inequality compare to what we know π to be?
  - *This inequality is quite accurate as* $3.10 < 3.14 < 3.21$*; there is only a difference of* $\frac{4}{100}$ *for the lower region and* $\frac{7}{100}$ *for the upper region.*

- We could continue the process of refining our estimate several more times to see that
$$3.14159 < \pi < 3.14160,$$
and then continue on to get an even more precise estimate of π. But at this point, it should be clear that we have a fairly good one already.
- We finish by making one more observation about π and irrational numbers in general. When we take the square of an irrational number such as π, we are doing it formally without exactly knowing the value of π^2. Since we can use a calculator to show that
$$3.14159 < \pi < 3.14160,$$
then, we also know that
$$3.14159^2 < \pi^2 < 3.14160^2$$
$$9.8695877281 < \pi\text{^}2 < 9.86965056.$$
Notice that the first 4 digits, 9.869, appear in the inequality. Therefore, we can say that $\pi^2 = 9.869$ is correct up to 3 decimal digits.

Exercises 1–4 (5 minutes)

Students work on Exercises 1–4 independently or in pairs. If necessary, model for students how to use the given decimal digits of the irrational number to *trap* the number in the inequality for Exercises 2–4. An online calculator was used to determine the decimal values of the squared numbers in Exercises 2–4. If handheld calculators are used, then the decimal values will be truncated to 8 places. However, this will not affect the estimate of the irrational numbers.

Exercises 1–4

1. **Gerald and Sarah are building a wheel with a radius of 6.5 cm and are trying to determine the circumference. Gerald says, "Because $6.5 \times 2 \times 3.14 = 40.82$, the circumference is 40.82 cm." Sarah says, "Because $6.5 \times 2 \times 3.10 = 40.3$ and $6.5 \times 2 \times 3.21 = 41.73$, the circumference is somewhere between 40.3 and 41.73." Explain the thinking of each student.**

   Gerald is using a common approximation for the number π to determine the circumference of the wheel. That is why he used 3.14 in his calculation. Sarah is using an interval between which the value of π falls, based on the work we did in class. We know that $3.10 < \pi < 3.21$; therefore, her calculations of the circumference uses numbers we know π to be between.

2. **Estimate the value of the irrational number $(6.12486\ldots)^2$.**

$$6.12486^2 < (6.12486\ldots)^2 < 6.12487^2$$
$$37.513\,910\,019\,6 < (6.12486\ldots)^2 < 37.514\,032\,516\,9$$

   $(6.12486\ldots)^2 = 37.51$ ***is correct up to two decimal digits.***

3. **Estimate the value of the irrational number $(9.204107\ldots)^2$.**

$$9.204\,107^2 < (9.204\,107\ldots)^2 < 9.204\,108^2$$
$$84.715\,585\,667\,449 < (9.204\,107\ldots)^2 < 84.715\,604\,075\,664$$

   $(9.204\,107\ldots)^2 = 84.715$ ***is correct up to three decimal digits.***

4. **Estimate the value of the irrational number $(4.014325...)^2$.**

$$4.014\,325^2 < (4.014\,325...)^2 < 4.014\,326^2$$
$$16.114\ 805\,205\,625 < (4.014\,325...)^2 < 16.114\,813\,242\,76$$

$(4.014\,325...)^2 = 16.1148$ *is correct up to four decimal digits.*

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- The area of a unit circle is π.
- We learned a method to estimate the value of π using graph paper, a unit circle, and areas.
- When we square the decimal expansion of an irrational number, we are doing it formally. This is similar to using approximation for computations. For that reason, we may only be accurate to a few decimal digits.

Lesson Summary

Irrational numbers, such as π, are frequently approximated in order to compute with them. Common approximations for π are 3.14 and $\frac{22}{7}$. It should be understood that using an approximate value of an irrational number for computations produces an answer that is accurate to only the first few decimal digits.

Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 14: Decimal Expansion of π

Exit Ticket

Describe how we found a decimal approximation for π.

Exit Ticket Sample Solutions

Describe how we found a decimal approximation for π.

To make our work easier, we looked at the number of unit squares in a quarter circle that comprised its area. We started by counting just the whole number of unit squares. Then, we continued to revise our estimate of the area by looking at parts of squares specifically to see if parts could be combined to make a whole unit square. We looked at the inside and outside boundaries and said that the value of π would be between these two numbers. The inside boundary is a conservative estimate of the value of π, and the outside boundary is an overestimate of the value of π. We could continue this process with smaller squares in order to refine our estimate for the decimal approximation of π.

Problem Set Sample Solutions

Students estimate the values of irrational numbers squared.

1. **Caitlin estimated π to be $3.10 < \pi < 3.21$. If she uses this approximation of π to determine the area of a circle with a radius of 5 cm, what could the area be?**

   The area of the circle with radius 5 cm will be between 77.5 cm^2 and 80.25 cm^2.

2. **Myka estimated the circumference of a circle with a radius of 4.5 in. to be 28.44 in. What approximate value of π did she use? Is it an acceptable approximation of π? Explain.**

$$C = 2\pi r$$
$$28.44 = 2\pi(4.5)$$
$$28.44 = 9\pi$$
$$\frac{28.44}{9} = \pi$$
$$3.16 = \pi$$

   Myka used 3.16 to approximate π. This is an acceptable approximation for π because it is in the interval that we approximated π to be in the lesson, $3.10 < \pi < 3.21$.

3. **A length of ribbon is being cut to decorate a cylindrical cookie jar. The ribbon must be cut to a length that stretches the length of the circumference of the jar. There is only enough ribbon to make one cut. When approximating π to calculate the circumference of the jar, which number in the interval $3.10 < \pi < 3.21$ should be used? Explain.**

   In order to make sure the ribbon is long enough, we should use an estimate of π that is closer to 3.21. We know that 3.10 is a fair estimate of π, but less than the actual value of π. Similarly, we know that 3.21 is a fair estimate of π but greater than the actual value of π. Since we can only make one cut, we should cut the ribbon so that there is a little more than we need, not less than. For that reason, an approximation of π closer to 3.21 should be used.

4. **Estimate the value of the irrational number $(1.86211\ldots)^2$.**

$$1.86211^2 < (1.86211\ldots)^2 < 1.86212^2$$
$$3.467\,453\,652\,1 < (1.86211\ldots)^2 < 3.467\,490\,894\,4$$

   $(1.86211\ldots)^2 = 3.4674$ is correct up to four decimal digits.

5. Estimate the value of the irrational number $(5.9035687\ldots)^2$.

$$5.903\,568\,7^2 < (5.903\,568\,7\ldots)^2 < 5.903\,568\,8^2$$
$$34.852\,123\,395\,619\,69 < (5.903\,568\,7\ldots)^2 < 34.852\,124\,576\,333\,44$$

$(5.903\,568\,7\ldots)^2 = 34.85212$ is correct up to five decimal digits.

6. Estimate the value of the irrational number $(12.30791\ldots)^2$.

$$12.30791^2 < (12.30791\ldots)^2 < 12.30792^2$$
$$151.484\,648\,568\,1 < (12.30791\ldots)^2 < 151.484\,894\,726\,4$$

$(12.30791\ldots)^2 = 151.484$ is correct up to three decimal digits.

7. Estimate the value of the irrational number $(0.6289731\ldots)^2$.

$$0.628\,973\,1^2 < (0.628\,973\,1\ldots)^2 < 0.628\,973\,2^2$$
$$0.395\,607\,160\,523\,61 < (0.628\,973\,1\ldots)^2 < 0.395\,607\,286\,3182\,4$$

$(0.6289731\ldots)^2 = 0.395607$ is correct up to six decimal digits.

8. Estimate the value of the irrational number $(1.112223333\ldots)^2$.

$$1.112\,223\,333^2 < (1.112\,223\,333\ldots)^2 < 1.112\,223\,334^2$$
$$1.237\,040\,742\,469\,628\,9 < (1.112\,223\,333\ldots)^2 < 1.237\,040\,744\,694\,075\,6$$

$(1.112\,223\,333\ldots)^2 = 1.237\,040\,74$ is correct up to eight decimal digits.

9. Which number is a better estimate for π, $\frac{22}{7}$ or 3.14? Explain.

Allow for both answers to be correct as long as the student provides a reasonable explanation.

Sample answer might be as follows.

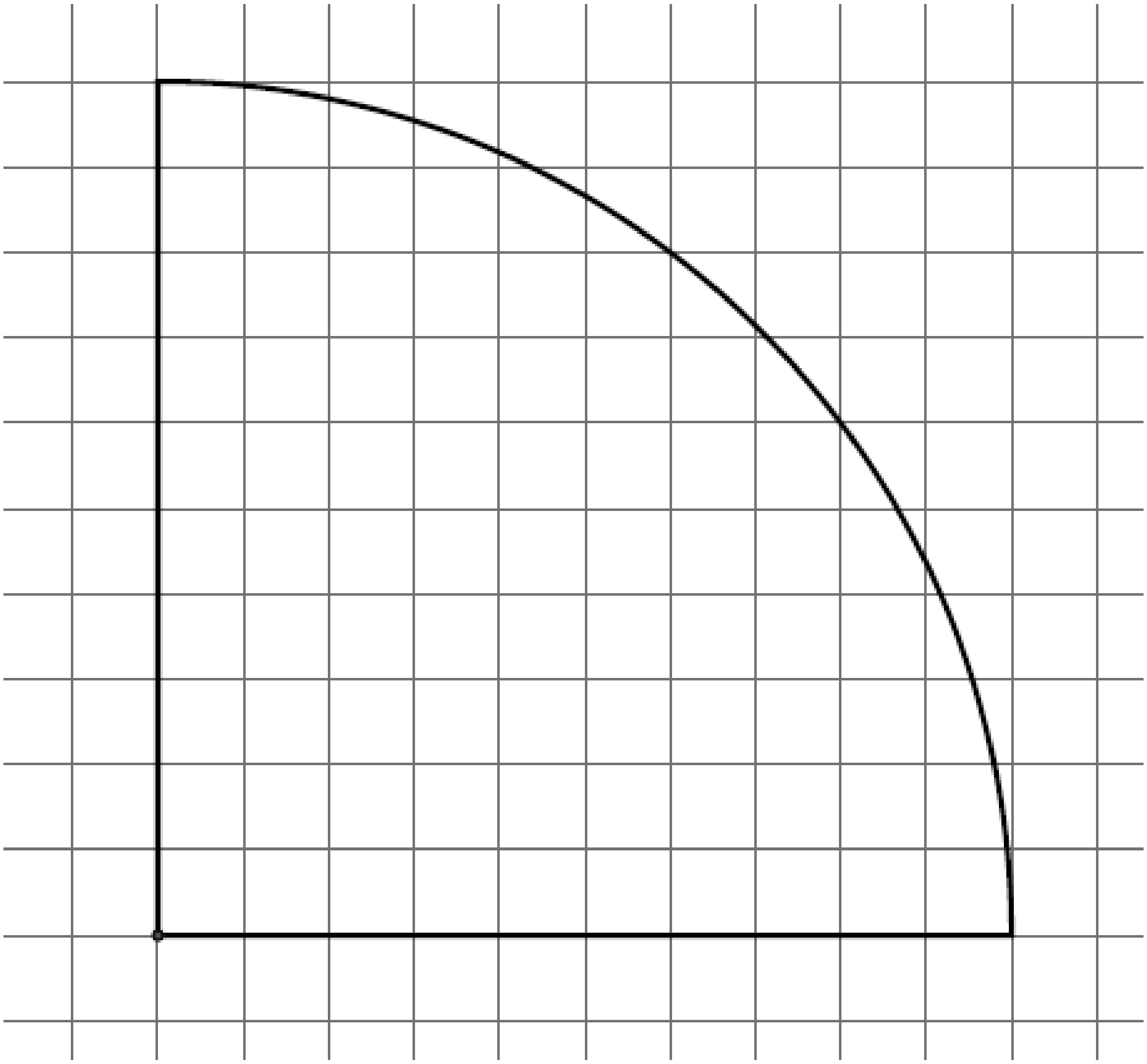
I think that 3.14 is a better estimate because when I find the decimal expansion, $\frac{22}{7} \approx 3.142857\ldots$; the number 3.14 is closer to the actual value of π.

10. To how many decimal digits can you correctly estimate the value of the irrational number $(4.56789012\ldots)^2$?

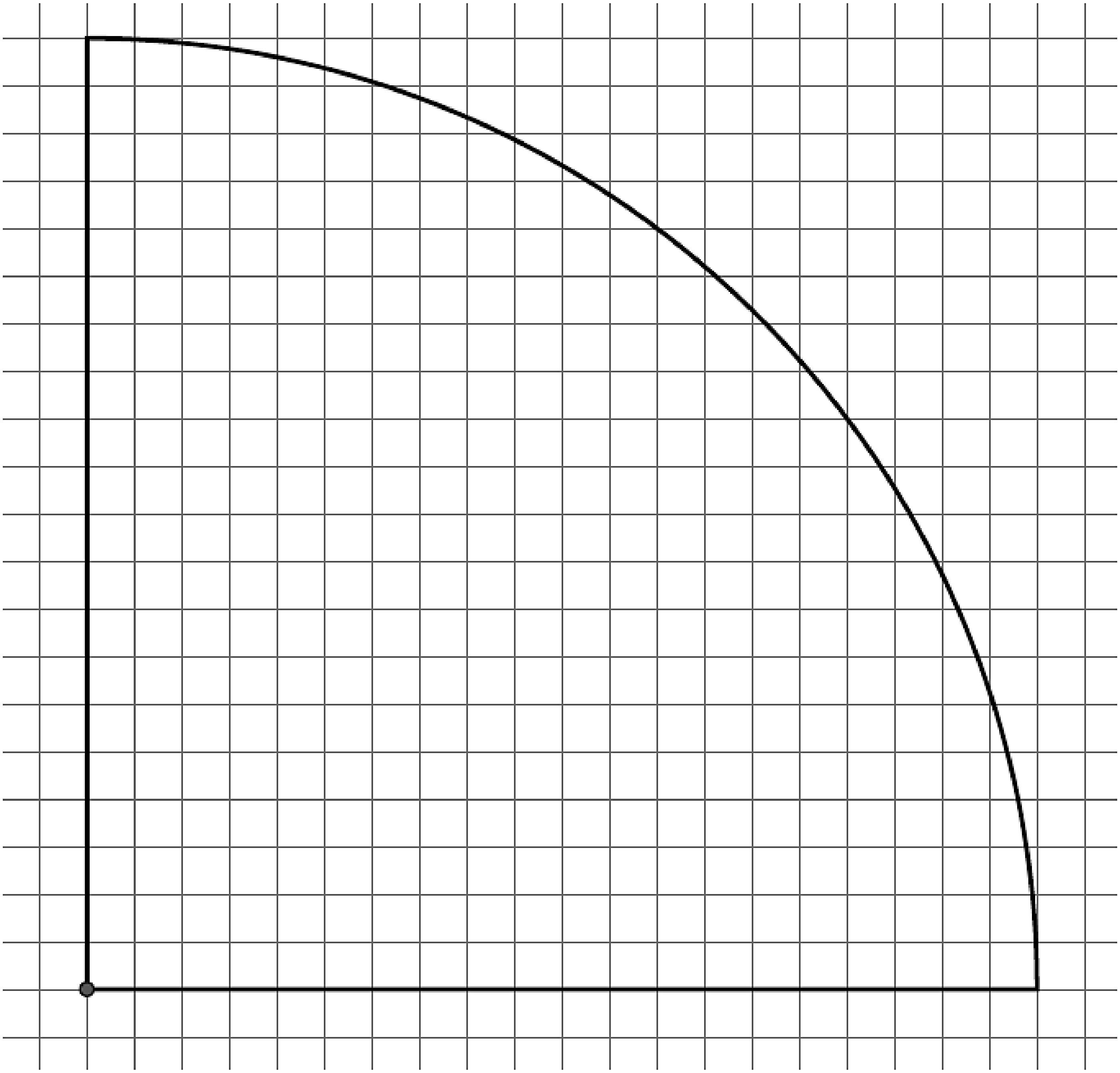
$$4.567\,890\,12^2 < (4.567\,890\,12\ldots)^2 < 4.567\,890\,13^2$$
$$20.865\,620\,148\,393\,614\,4 < (1.112\,223\,333\ldots)^2 < 20.865\,620\,239\,751\,416\,9$$

$(4.567\,890\,12\ldots)^2 = 20.865\,620$ is correct up to six decimal digits.

10 by 10 Grid



20 by 20 Grid



Name ____ Date ____________________

1.

a. What is the decimal expansion of the number $\frac{35}{7}$? Is the number $\frac{35}{7}$ rational or irrational? Explain.

b. What is the decimal expansion of the number $\frac{4}{33}$? Is the number $\frac{4}{33}$ rational or irrational? Explain.

2.

a. Write $0.\overline{345}$ as a fraction.

b. Write $2.8\overline{40}$ as a fraction.

c. Brandon stated that 0.66 and $\frac{2}{3}$ are equivalent. Do you agree? Explain why or why not.

d. Between which two positive integers does $\sqrt{33}$ lie?

e. For what integer x is $\sqrt{x}$ closest to 5.25? Explain.

3. Identify each of the following numbers as rational or irrational. If the number is irrational, explain how you know.

   a. $\sqrt{29}$

   b. $5.\overline{39}$

   c. $\frac{12}{4}$

   d. $\sqrt{36}$

   e. $\sqrt{5}$

   f. $\sqrt[3]{27}$

   g. $\pi = 3.141592 \ldots$

   h. Order the numbers in parts (a)–(g) from least to greatest, and place on a number line.

4. Circle the greater number in each of the pairs (a)–(e) below.

   a. Which is greater? 8 or $\sqrt{60}$

   b. Which is greater? 4 or $\sqrt{26}$

   c. Which is greater? $\sqrt[3]{64}$ or $\sqrt{16}$

   d. Which is greater? $\sqrt[3]{125}$ or $\sqrt{30}$

   e. Which is greater? -7 or $-\sqrt{42}$

   f. Put the numbers 9, $\sqrt{52}$, and $\sqrt[3]{216}$ in order from least to greatest. Explain how you know which order to put them in.

5.

a. Between which two labeled points on the number line would $\sqrt{5}$ be located?

b. Explain how you know where to place $\sqrt{5}$ on the number line.

c. How could you improve the accuracy of your estimate?

6. Determine the positive solution for each of the following equations.

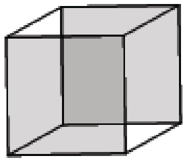
   a. $121 = x^2$

   b. $x^3 = 1000$

   c. $17 + x^2 = 42$

   d. $x^3 + 3x - 9 = x - 1 + 2x$

e. The cube shown has a volume of 216 cm^3.



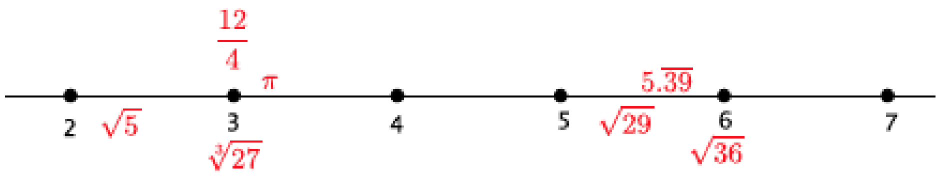
$V = 216\text{ cm}^3$

i. Write an equation that could be used to determine the length, l, of one side.

ii. Solve the equation, and explain how you solved it.

A Progression Toward Mastery

Assessment Task Item		**STEP 1** **Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.**	**STEP 2** **Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.**	**STEP 3** **A correct answer with some evidence of reasoning or application of mathematics to solve the problem, OR an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.**	**STEP 4** **A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.**
1	**a–b** **8.NS.A.1**	Student makes little or no attempt to respond to either part of the problem. OR Student answers both parts incorrectly.	Student identifies one or both numbers as rational. Student may not write the decimal expansions of the numbers and does not reference the decimal expansions of the numbers in his or her explanation.	Student identifies both numbers as rational. Student correctly writes the decimal expansion of each number. Student may not reference the decimal expansion in his or her explanation but uses another explanation (e.g., the numbers are quotients of integers).	Student identifies both numbers as rational. Student correctly writes the decimal expansion of $\frac{35}{7}$ as $5.000\ldots$, or 5, and $\frac{4}{33}$ as $0.121212\ldots$, or $0.\overline{12}$. Student explains that the numbers are rational by stating that every rational number has a decimal expansion that repeats eventually. Student references the decimal expansion of $\frac{35}{7}$ with the repeating decimal of zero and the decimal expansion of $\frac{4}{33}$ with the repeating decimal of 12.
2	**a–b** **8.NS.A.1**	Student does not attempt problem or writes answers that are incorrect for both parts.	Student is able to write one of the parts (a)–(b) correctly as a fraction. OR Student answers both parts incorrectly but shows some evidence of understanding how to convert an infinite, repeating decimal to a fraction.	Student is able to write both parts (a)–(b) correctly as fractions. OR Student writes one part correctly but makes computational errors leading to an incorrect answer for the other part.	Student correctly writes both parts (a)–(b) as fractions. Part (a) is written as $0.\overline{345} = \frac{115}{333}$ (or equivalent), and part (b) is written as $2.8\overline{40} = \frac{2812}{990}$ (or equivalent).

	c **8.NS.A.1**	Student agrees with Brandon or writes an explanation unrelated to the problem.	Student does not agree with Brandon. Student writes a weak explanation defending his position.	Student does not agree with Brandon. Student writes an explanation that shows why the equivalence was incorrect reasoning that 0.66 does not equal $\frac{2}{3}$ or that $\frac{2}{3}$ does not equal 0.66 but fails to include both explanations.	Student does not agree with Brandon. Student writes an explanation that shows 0.66 does not equal $\frac{2}{3}$ and that $\frac{2}{3}$ does not equal 0.66.
	d–e **8.NS.A.2**	Student does not attempt problem or writes answers that are incorrect for both parts (d)–(e).	Student is able to answer at least one of the parts (d)–(e) correctly. Student may or may not provide a weak justification for answer selection.	Student is able to answer both parts (d)–(e) correctly. Student may or may not provide a justification for answer selection. Explanation includes some evidence of mathematical reasoning.	Student correctly answers both parts (d)–(e); for part (d), $\sqrt{33}$ is between positive integers 5 and 6; for part (e), $x \approx 28$. Student provides an explanation that included solid reasoning related to rational approximation.
3	**a–f** **8.NS.A.1** **8.EE.A.2**	Student does not attempt problem or writes correct answers for one or two parts of (a)–(g).	Student correctly identifies three or four parts of (a)–(g) as rational or irrational. Student may or may not provide a weak explanation for those numbers that are irrational.	Student correctly identifies five or six parts of (a)–(g) correctly as rational or irrational. Student may provide an explanation for those numbers that are irrational but does not refer to their decimal expansion or any other mathematical reason.	Student correctly identifies all seven parts of (a)–(g); (a) irrational, (b) rational, (c) rational, (d) rational, (e) irrational, (f) rational, and (g) irrational. Student explains parts (a), (e), and (g) as irrational by referring to their decimal expansion or the fact that the radicand was not a perfect square.
	h **8.NS.A.2**	Student correctly places zero to two numbers correctly on the number line.	Student correctly places three or four of the numbers on the number line.	Student correctly places five of the six numbers on the number line.	Student correctly places all six numbers on the number line. (Correct answers noted in red below.)
					

4	**a–e** **8.NS.A.2** **8.EE.A.2**	Student correctly identifies the larger number zero to one time in parts (a)–(e).	Student correctly identifies the larger number two to three times in parts (a)–(e).	Student correctly identifies the larger number four times in parts (a)–(e).	Student correctly identifies the larger number in all of parts (a)–(e); (a) 8, (b) $\sqrt{26}$, (c) numbers are equal, (d) $\sqrt{30}$, and (e) $-\sqrt{42}$.
	f **8.NS.A.2** **8.EE.A.2**	Student does not attempt the problem or responds incorrectly. Student does not provide an explanation.	Student may correctly order the numbers from greatest to least. Student may or may not provide a weak explanation for how he put the numbers in order.	Student correctly orders the numbers from least to greatest. Student provides a weak explanation for how he put the numbers in order.	Student correctly orders the numbers from least to greatest: $\sqrt[3]{216}$, $\sqrt{52}$, 9. Explanation includes correct mathematical vocabulary (e.g., square root, cube root, between perfect squares).
5	**a–c** **8.NS.A.2**	Student makes little or no attempt to do the problem. OR Student may or may not place $\sqrt{5}$ correctly on the number line for part (a). For parts (b)–(c), student does not provide an explanation.	Student may or may not have placed $\sqrt{5}$ correctly on the number line for part (a). For parts (b)–(c), student may or may not provide a weak explanation for how the number was placed. Student may or may not provide a weak explanation for how to improve accuracy of approximation.	Student places $\sqrt{5}$ correctly on the number line for part (a). For parts (b)–(c), student may provide a weak explanation for how the number was placed. Student may provide a weak explanation for how to improve accuracy of approximation. Student references the method of rational approximation.	Student correctly places $\sqrt{5}$ between 2.2 and 2.3 on the number line for part (a). For parts (b)–(c), student explains the method of rational approximation to locate the approximate position of $\sqrt{5}$ on the number line. Student explains how to continue the rational approximation to include increasing smaller intervals to improve the accuracy of the estimate.
6	**a–b** **8.EE.A.2**	Student makes little or no attempt to solve either equation or writes incorrect answers for both.	Student writes the correct answer for one of the equations but does not write the answer in the appropriate form (i.e., 11 or 10 instead of $x = 11$ or $x = 10$).	Student solves at least one of the equations correctly or solves both correctly but does not write the answer in the appropriate form (i.e., 11 and 10 instead of $x = 11$ and $x = 10$).	Student solves both equations correctly for parts (a)–(b); (a) $x = 11$, and (b) $x = 10$.
	c–d **8.EE.A.2**	Student makes little or no attempt to solve either equation or writes incorrect answers for both.	Student may solve one equation correctly. OR Student uses properties of rational numbers to transform the equations but cannot determine the correct value of x or makes computational errors leading to incorrect solutions for x.	Student solves one of the equations correctly but makes computational errors leading to an incorrect answer for the other equation.	Student solves both equations correctly for parts (c)–(d); (c) $x = 5$, and (d) $x = 2$.

	e **8.EE.A.2**	Student makes little or no attempt to solve the problem.	Student may state that the length of one side of the cube is 6 cm but does not write an equation or solve it.	Student states that the length of one side of the cube is 6 cm. Student correctly writes and solves an equation but provides a weak explanation for solving it.	Student states that the length of one side of the cube is 6 cm. Student correctly writes and solves the equation. Student provides a clear and complete explanation for solving the equation that includes some reference to cube roots and why $\sqrt[3]{216} = 6$.

Name ______________________________ Date ______________

1.

a. What is the decimal expansion of the number $\frac{35}{7}$? Is the number $\frac{35}{7}$ rational or irrational? Explain.

$$\frac{35}{7} = 5.000\ldots$$

The number $\frac{35}{7}$ is a rational number. Rational numbers have decimal expansions that repeat. In this case, the decimal that repeats is a zero.

b. What is the decimal expansion of the number $\frac{4}{33}$? Is the number $\frac{4}{33}$ rational or irrational? Explain.

$$\begin{array}{r} 0.1212 \\ 33\overline{)4.00000} \\ \underline{33} \\ 70 \\ \underline{66} \\ 40 \\ \underline{33} \\ 70 \\ \underline{66} \\ 4 \end{array}$$

$$\frac{4}{33} = 0.1212\ldots = 0.\overline{12}$$

The number $\frac{4}{33}$ is rational. Numbers that have decimal expansions that repeat. The digits 12 repeat in the decimal expansion of $\frac{4}{33}$. So, $\frac{4}{33}$ is rational.

2.

a. Write $0.\overline{345}$ as a fraction.

Let x be $0.\overline{345}$.

$$10^3 x = 10^3(0.\overline{345})$$
$$1000x = 1000(0.\overline{345})$$
$$1000x = 345.\overline{345}$$
$$1000x = 345 + x$$
$$1000x - x = 345 + x - x$$
$$999x = 345$$
$$x = \frac{345}{999} = \frac{115}{333}$$

b. Write $2.8\overline{40}$ as a fraction.

Let x be $2.8\overline{40}$.

$$10x = 28.\overline{40}$$
$$10x = 28 + 0.\overline{40}$$
$$10x = 28 + \frac{40}{99}$$
$$10x = \frac{(28)(99) + 40}{99}$$
$$x = \frac{2812}{99}\left(\frac{1}{10}\right)$$
$$x = \frac{2812}{990} = \frac{1406}{495}$$

Let y be $0.\overline{40}$.

$$10^2 y = 10^2(0.\overline{40})$$
$$100y = 40.\overline{40}$$
$$100y = 40 + y$$
$$100y - y = 40 + y - y$$
$$99y = 40$$
$$y = \frac{40}{99}$$

$$2.8\overline{40} = \frac{2812}{990} = \frac{1406}{495}$$

c. Brandon stated that 0.66 and $\frac{2}{3}$ are equivalent. Do you agree? Explain why or why not.

No, I do not agree with Brandon. The decimal 0.66 is equal to the fraction $\frac{66}{100} = \frac{33}{50}$, not $\frac{2}{3}$. Also, the number $\frac{2}{3}$ is equal to the infinite decimal $0.\overline{6}$. The number 0.66 is a finite decimal. Therefore, 0.66 and $\frac{2}{3}$ are not equivalent.

d. Between which two positive integers does $\sqrt{33}$ lie?

The number $\sqrt{33}$ is between 5 and 6 because

$5^2 < (\sqrt{33})^2 < 6^2$.

e. For what integer x is $\sqrt{x}$ closest to 5.25? Explain.

$(5.25)^2 = 27.5625$

Since $\sqrt{x}$ is the square root of x, then x^2 will give me the integer that belongs in the square root. $(5.25)^2 = 27.5625$, which is closest to the integer 28.

3. Identify each of the following numbers as rational or irrational. If the number is irrational, explain how you know.

   a. $\sqrt{29}$ Irrational because 29 is not a perfect square and $\sqrt{29}$ has an infinite decimal expansion that does not repeat.

   b. $5.\overline{39}$ Rational

   c. $\frac{12}{4}$ Rational

   d. $\sqrt{36}$ Rational

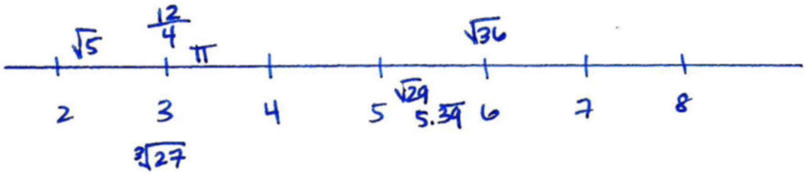
   e. $\sqrt{5}$ Irrational because 5 is not a perfect square and $\sqrt{5}$ has an infinite decimal expansion that does not repeat.

   f. $\sqrt[3]{27}$ Rational

   g. $\pi = 3.141592\ldots$ Irrational because pi has a decimal expansion that does not repeat.

   h. Order the numbers in parts (a)–(g) from least to greatest, and place on a number line.

   $\sqrt{29}$: $5^2 < (\sqrt{29})^2 < 6^2$, $5.3^2 < (\sqrt{29})^2 < 5.4^2$, $5.38^2 < (\sqrt{29})^2 < 5.39^2$



4. Circle the greater number in each of the pairs (a)–(e) below.

   a. Which is greater? [✓ 8] or $\sqrt{60}$

   b. Which is greater? 4 or [✓ $\sqrt{26}$]

   c. Which is greater? $\sqrt[3]{64}$ or $\sqrt{16}$

      The numbers are equal: $\sqrt[3]{64} = 4$, $\sqrt{16} = 4$.

   d. Which is greater? $\sqrt[3]{125}$ or [✓ $\sqrt{30}$]

   e. Which is greater? -7 or [✓ $-\sqrt{42}$]

   f. Put the numbers 9, $\sqrt{52}$, and $\sqrt[3]{216}$ in order from least to greatest. Explain how you know which order to put them in.

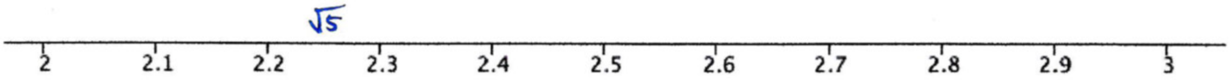
      $\sqrt{52}$ is between 7 and 8.

      $\sqrt[3]{216} = 6$

      In order from least to greatest:

      $\sqrt[3]{216}$, $\sqrt{52}$, 9

5.



a. Between which two labeled points on the number line would $\sqrt{5}$ be located?

The number $\sqrt{5}$ is between 2.2 and 2.3.

b. Explain how you know where to place $\sqrt{5}$ on the number line.

I knew that $\sqrt{5}$ was between 2 and 3 but closer to 2. So next, I checked intervals of tenths beginning with 2.0 to 2.1. The interval that $\sqrt{5}$ fit between was 2.2 and 2.3. because

$$2.2^2 < (\sqrt{5})^2 < 2.3^2, \quad 4.84 < 5 < 5.29.$$

c. How could you improve the accuracy of your estimate?

To improve the estimate, I would have to continue with the method of rational approximation to determine which interval of hundredths $\sqrt{5}$ fits between. Once I knew the interval of hundredths, I would check the interval of thousandths, and so on.

$$\sqrt{5}: \; 2^2 < (\sqrt{5})^2 < 3^2, \quad 2.2^2 < (\sqrt{5})^2 < 2.3^2$$
$$4 < 5 < 9 \qquad 4.84 < 5 < 5.29$$

6. Determine the positive solution for each of the following equations.

a. $121 = x^2$

$\sqrt{121} = \sqrt{x^2}$

$11 = x$

$11^2 = 121$

$121 = 121$

b. $x^3 = 1000$

$\sqrt[3]{x^3} = \sqrt[3]{1000}$

$x = 10$

$10^3 = 1000$

$1000 = 1000$

c. $17 + x^2 = 42$

$17 - 17 x^2 = 42 - 17$

$x^2 = 25$

$\sqrt{x^2} = \sqrt{25}$

$x = 5$

$17 + 5^2 = 42$

$17 + 25 = 42$

$42 = 42$

d. $x^3 + 3x - 9 = x - 1 + 2x$

$x^3 + 3x - 3x - 9 = x - 1 + 2x - 3x$

$x^3 - 9 = -1$

$x^3 - 9 + 9 = -1 + 9$

$x^3 = 8$

$\sqrt[3]{x^3} = \sqrt[3]{8}$

$x = 2$

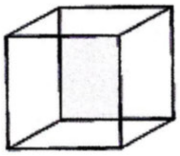
$2^3 + (3)(2) - 9 = 2 - 1 + (2)(2)$

$8 + 6 - 9 = 2 - 1 + 4$

$14 - 9 = 1 + 4$

$5 = 5$

e. The cube shown has a volume of 216 cm^3.



$V = 216\text{ cm}^3$

i. Write an equation that could be used to determine the length, l, of one side.

$V = l^3$

$216 = l^3$

ii. Solve the equation, and explain how you solved it.

$216 = l^3$

$\sqrt[3]{216} = \sqrt[3]{l^3}$

$6 = l$

The length of one side is 6cm.

To solve the equation, I had to take the cube root of both sides of the equation. The cube root of l^3, $\sqrt[3]{l^3}$, is l. The cube root of 216, $\sqrt[3]{216}$, is 6 because $6^3 = 216$. Therefore, the length of one side of the cube is 6cm.

Topic C:

The Pythagorean Theorem

8.G.B.6, 8.G.B.7, 8.G.B.8

Focus Standards:	8.G.B.6	Explain a proof of the Pythagorean Theorem and its converse.
	8.G.B.7	Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
	8.G.B.8	Apply the Pythagorean Theorem to find the distance between two points in a coordinate system.
Instructional Days:	4	
Lesson 15:	Pythagorean Theorem, Revisited (S)[1]	
Lesson 16:	Converse of the Pythagorean Theorem (S)	
Lesson 17:	Distance on the Coordinate Plane (P)	
Lesson 18:	Applications of the Pythagorean Theorem (E)	

In Lesson 15, students engage with another proof of the Pythagorean theorem. This time, students compare the areas of squares that are constructed along each side of a right triangle in conjunction with what they know about similar triangles. Now that students know about square roots, students can determine the approximate length of an unknown side of a right triangle even when the length is not a whole number. Lesson 16 shows students another proof of the converse of the Pythagorean theorem based on the notion of congruence. Students practice explaining proofs in their own words in Lessons 15 and 16 and apply the converse of the theorem to make informal arguments about triangles as right triangles. Lesson 17 focuses on the application of the Pythagorean theorem to calculate the distance between two points on the coordinate plane. Lesson 18 gives students practice applying the Pythagorean theorem in a variety of mathematical and real-world scenarios. Students determine the height of isosceles triangles, determine the length of the diagonal of a rectangle, and compare lengths of paths around a right triangle.

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[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

Lesson 15: Pythagorean Theorem, Revisited

Student Outcomes

- Students know that the Pythagorean theorem can be interpreted as a statement about the areas of similar geometric figures constructed on the sides of a right triangle.
- Students explain a proof of the Pythagorean theorem.

Lesson Notes

The purpose of this lesson is for students to review and practice presenting the proof of the Pythagorean theorem using similar triangles. Then, students will apply this knowledge to another proof that uses areas of similar figures such as squares.

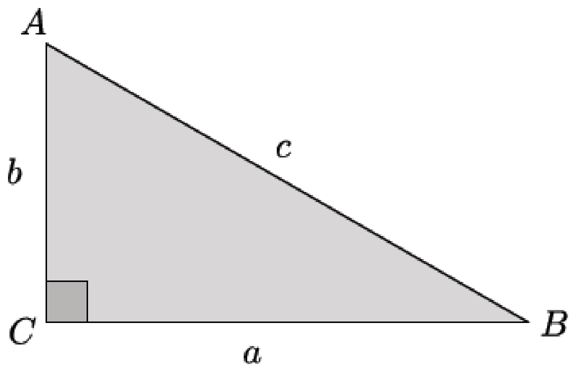
Classwork

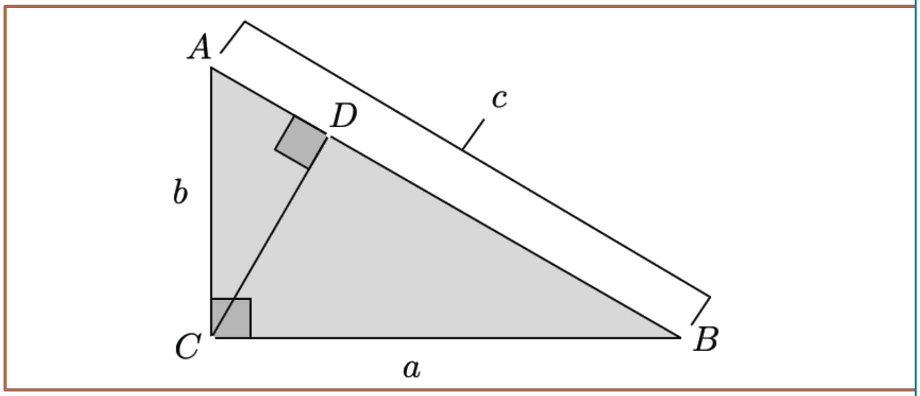
Discussion (20 minutes)

This discussion is an opportunity for students to practice explaining a proof of the Pythagorean theorem using similar triangles. Instead of leading the discussion, consider posing the questions, one at a time, to small groups of students and allow time for discussions. Then, have select students share their reasoning while others critique.

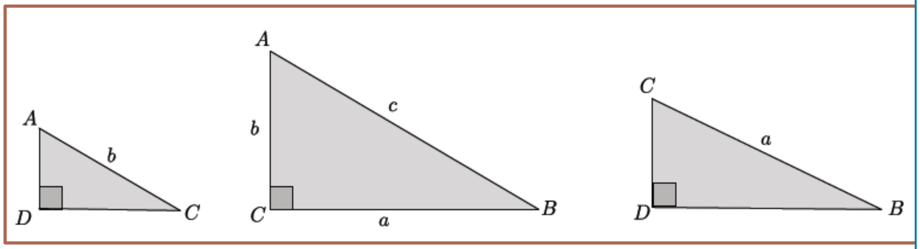
- To prove the Pythagorean theorem, $a^2 + b^2 = c^2$, use a right triangle, shown below. Begin by drawing a segment from the right angle, perpendicular to side AB through point C. Label the intersection of the segments point D.

Proof of the Pythagorean Theorem





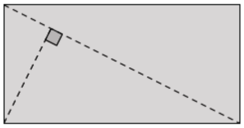
- Using one right triangle, we created 3 right triangles. Name those triangles.
    - *The three triangles are* $\triangle ABC$, $\triangle ACD$, *and* $\triangle BCD$.
- We can use our basic rigid motions to reorient the triangles so they are easier to compare, as shown below.



- The next step is to show that these triangles are similar. Begin by showing that $\triangle ADC \backsim \triangle ACB$. Discuss in your group.
    - $\triangle ADC$ *and* $\triangle ACB$ *are similar because they each have a right angle, and they each share* $\angle A$. *Then, by the AA criterion for similarity,* $\triangle ADC \backsim \triangle ACB$.
- Now, show that $\triangle ACB \backsim \triangle CDB$. Discuss in your group.
    - $\triangle ACB \backsim \triangle CDB$ *because they each have a right angle, and they each share* $\angle B$. *Then, by the AA criterion for similarity,* $\triangle ACB \backsim \triangle CDB$.
- Are $\triangle ADC$ and $\triangle CDB$ similar? Discuss in your group.
    - *We know that similarity has the property of transitivity; therefore, since* $\triangle ADC \backsim \triangle ACB$, *and* $\triangle ACB \backsim \triangle CDB$, *then* $\triangle ADC \backsim \triangle CDB$.

Scaffolding:

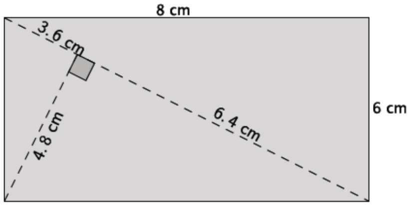
- A good hands-on visual that can be used here requires a 3×5 notecard. Have students draw the diagonal and then draw the perpendicular line from C to side AB.



- Make sure students label all of the parts to match the triangle to the left. Next, have students cut out the three triangles. Students will then have a notecard version of the three triangles shown below and can use them to demonstrate the similarity among them.
- The next scaffolding box shows a similar diagram for the concrete case of a 6-8-10 triangle.

Scaffolding:

You may also consider showing a concrete example, such as a 6-8-10 triangle, along with the general proof.



You can have students verify similarity using a protractor to compare corresponding angle measures. There is a reproducible available at the end of the lesson.

- If we consider $\triangle ADC$ and $\triangle ACB$, we can write a statement about corresponding sides being equal in ratio that will help us reach our goal of showing $a^2 + b^2 = c^2$. Discuss in your group.
    - *Using $\triangle ADC$ and $\triangle ACB$, we can write*

      $$\frac{|AC|}{|AB|} = \frac{|AD|}{|AC|}$$

      which is equal to

      $$|AC|^2 = |AB| \cdot |AD|.$$

      Since $|AC| = b$, we have

      $$b^2 = |AB| \cdot |AD|.$$
- Consider that $\triangle ACB$ and $\triangle CDB$ will give us another piece that we need. Discuss in your group.
    - *Using $\triangle ACB$ and $\triangle CDB$, we can write*

      $$\frac{|BA|}{|BC|} = \frac{|BC|}{|BD|}$$

      which is equal to

      $$|BC|^2 = |BA| \cdot |BD|.$$

      Since $|BC| = a$, we have

      $$a^2 = |BA| \cdot |BD|.$$
- The two equations $b^2 = |AB| \cdot |AD|$ and $a^2 = |BA| \cdot |BD|$ are all that we need to finish the proof. Discuss in your group.
    - *By adding the equations together, we have*

      $$a^2 + b^2 = |AB| \cdot |AD| + |BA| \cdot |BD|.$$

      The length $|AB| = |BA| = c$, so by substitution we have

      $$a^2 + b^2 = c \cdot |AD| + c \cdot |BD|.$$

      Using the distributive property

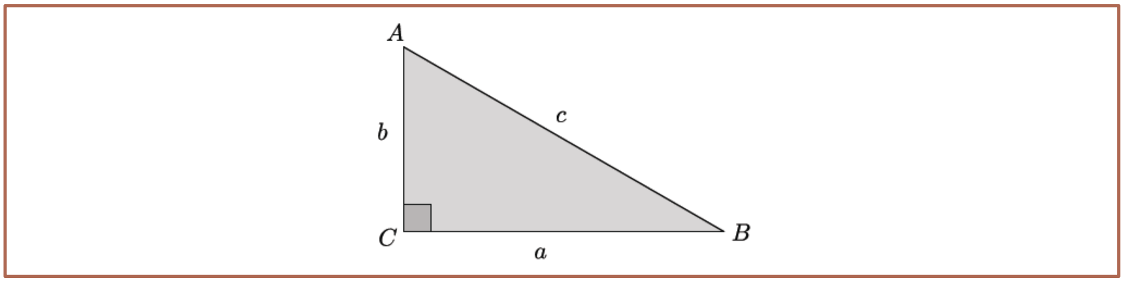
      $$a^2 + b^2 = c \cdot (|AD| + |BD|).$$

      The length $|AD| + |BD| = c$, so by substitution

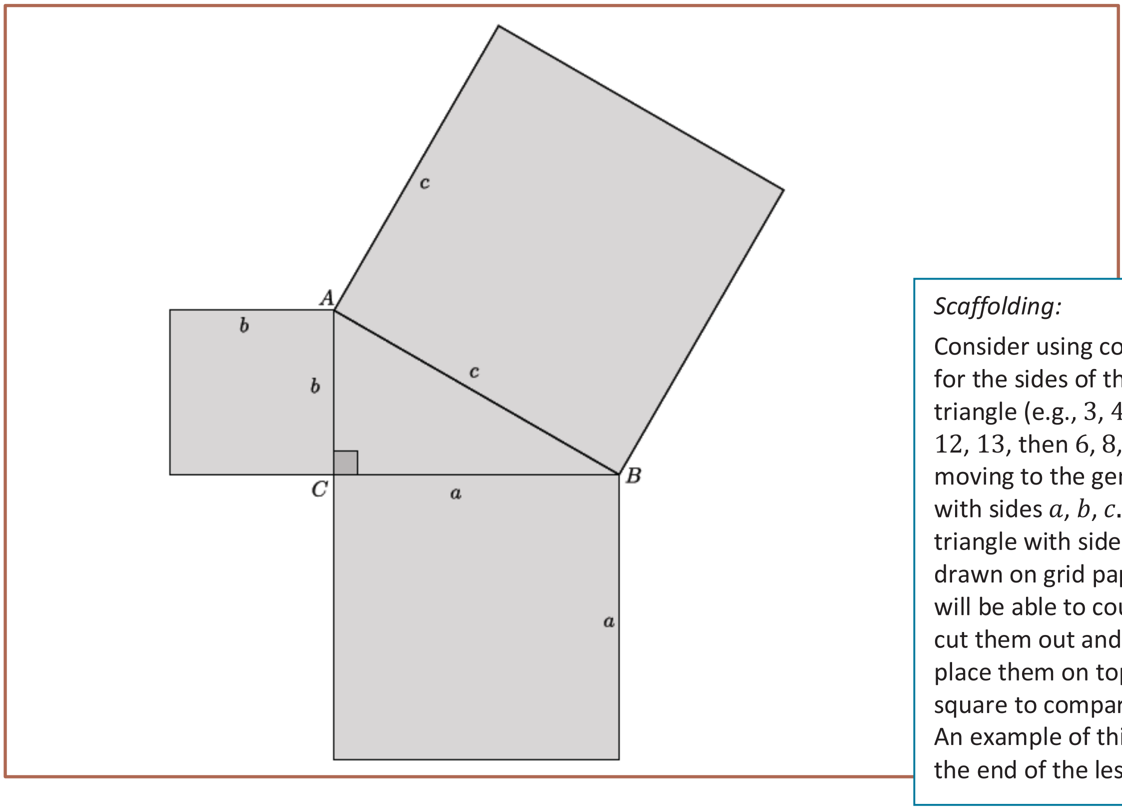
      $$a^2 + b^2 = c \cdot c$$
      $$a^2 + b^2 = c^2.$$

Discussion (15 minutes)

- Now, let's apply this knowledge to another proof of the Pythagorean theorem. Compare the area of similar figures drawn from each side of a right triangle. We begin with a right triangle:



- Next, we will construct squares off of each side of the right triangle in order to compare the areas of *similar* figures. However, are all squares similar? Discuss in your group.
  - *Yes, all squares are similar. Assume you have a square with side length equal to* 1 *unit. You can dilate from a center by any scale factor to make a square of any size similar to the original one.*



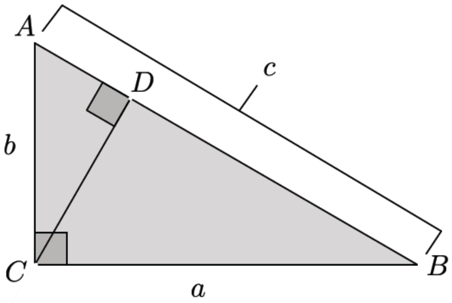
Scaffolding:

Consider using concrete values for the sides of the right triangle (e.g., 3, 4, 5, then 5, 12, 13, then 6, 8, 10) and then moving to the general triangle with sides a, b, c. Given a triangle with sides 3, 4, 5 drawn on grid paper, students will be able to count squares or cut them out and physically place them on top of the larger square to compare the areas. An example of this is shown at the end of the lesson.

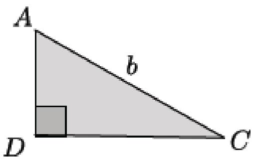
- What would it mean, geometrically, for $a^2 + b^2$ to equal c^2?
  - *It means that the sum of the areas of* a^2 *and* b^2 *is equal to the area* c^2.

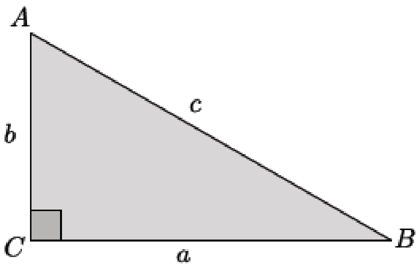
There are two possible ways to continue; one way is by examining special cases on grid paper, as mentioned in the scaffolding box above, and showing the relationship between the squares physically. The other way is by using the algebraic proof of the general case that continues below.

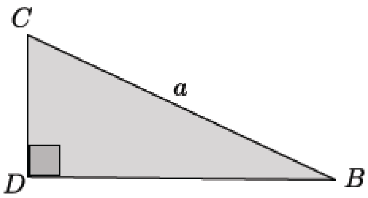
- This is where the proof using similar triangles will be helpful.



- When we compared $\triangle ACB$ and $\triangle CDB$, we wrote a statement about their corresponding side lengths, $\frac{|BA|}{|BC|} = \frac{|BC|}{|BD|}$, leading us to state that $|BC|^2 = |BA| \cdot |BD|$ and $a^2 = |BA| \cdot |BD|$. How might this information be helpful in leading us to show that the areas of $a^2 + b^2$ are equal to the area of c^2? Discuss in your group.



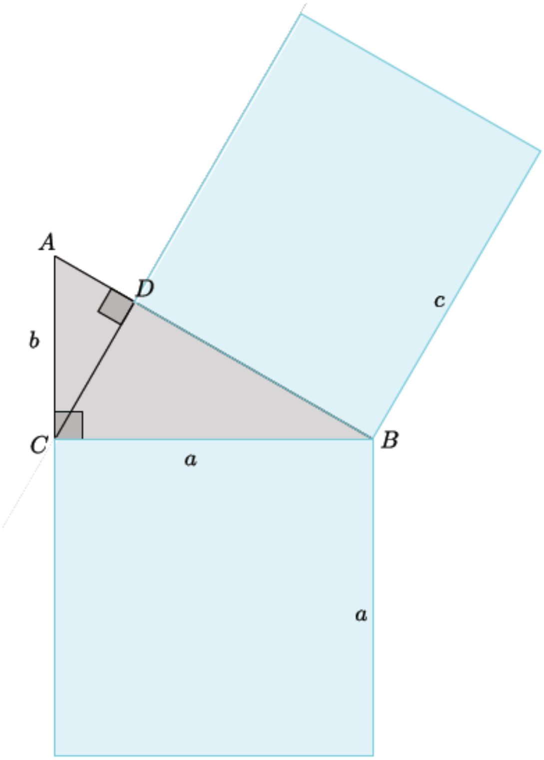




  - *Since $|BA| = c$, then we have $a^2 = c \cdot |BD|$, which is part of the area of c^2 that we need.*

MP.2

- Explain the statement $a^2 = c \cdot |BD|$ in terms of the diagram below.



- The square built from the leg of length a is equal in area to the rectangle built from segment BD, with length c. This is part of the area of the square with side c.

- Now, we must do something similar with the area of b^2. Discuss in your group.
    - *Using $\triangle ADC$ and $\triangle ACB$, we wrote*

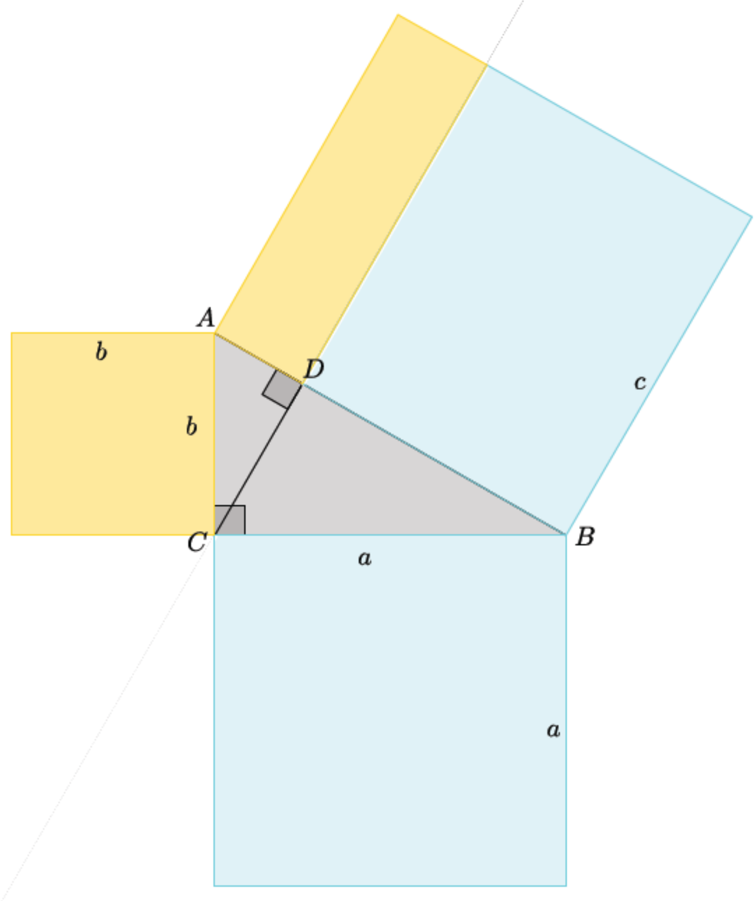
      $$\frac{|AC|}{|AB|} = \frac{|AD|}{|AC|},$$

      which is equal to

      $$|AC|^2 = |AB| \cdot |AD|.$$

      By substitution

      $$b^2 = |AB| \cdot |AD|$$
      $$b^2 = c \cdot |AD|.$$
- Explain the statement $b^2 = c \cdot |AD|$ in terms of the diagram below.

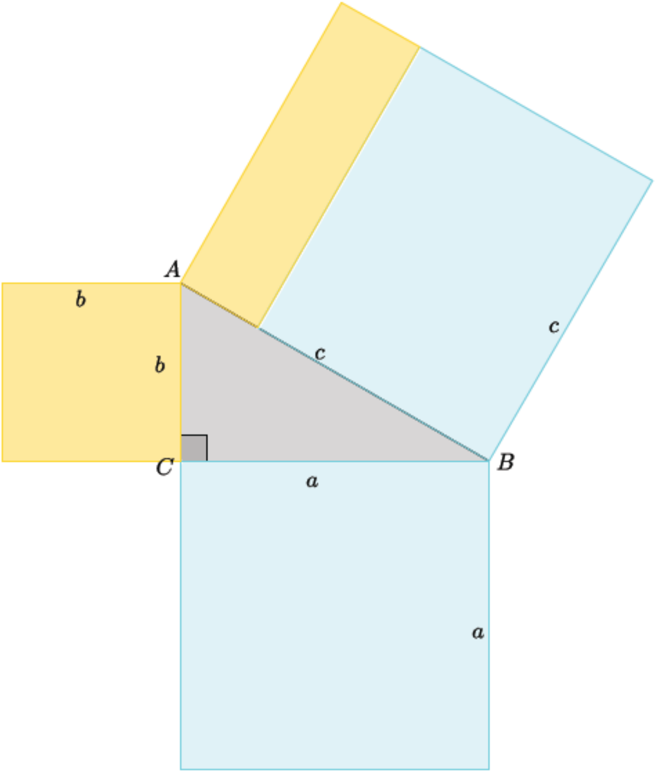


MP.2

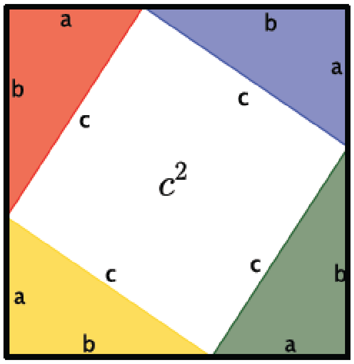
- *The square built from the leg of length b is equal, in area, to the rectangle built from segment AD, with length c. This is the other part of the area of the square with side c.*
- Our knowledge of similar figures, as well as our understanding of the proof of the Pythagorean theorem using similar triangles, led us to another proof where we compared the areas of similar figures constructed off the sides of a right triangle. In doing so, we have shown that $a^2 + b^2 = c^2$, in terms of areas.

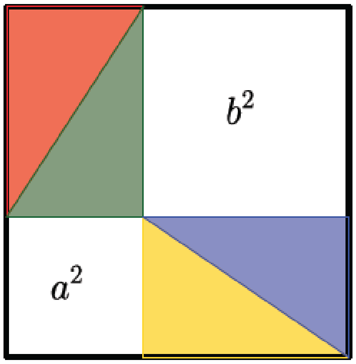
- Explain how the diagram shows that the Pythagorean theorem is true.
  - *The Pythagorean theorem states that given a right triangle with lengths a, b, c that $a^2 + b^2 = c^2$. The diagram shows that the area of the squares off of the legs a and b are equal to the area off of the hypotenuse c. Since the area of a square is found by multiplying a side by itself, then the area of a square with length a is a^2, b is b^2, and c is c^2. The diagram shows that the areas $a^2 + b^2$ are equal to the area of c^2, which is exactly what the theorem states.*

MP.2



To solidify student understanding of the proof, consider showing the six-minute video to students located at http://www.youtube.com/watch?v=QCyvxYLFSfU. If you have access to multiple computers or tablets, have small groups of students watch the video together, so they can pause and replay parts of the proof as necessary.





$$c^2 = a^2 + b^2$$

Scaffolding:

The geometric illustration of the proof, shown to the left, can be used as further support or as an extension to the claim that the sum of the areas of the smaller squares is equal to the area of the larger square.

Another short video that demonstrates $a^2 + b^2 = c^2$ using area is at the following link: http://9gag.com/gag/aOqPoMD/cool-demonstration-of-the-pythagorean-theorem.

It does not *explain* the proof but merely shows that it is true.

Closing (5 minutes)

Consider having students explain how to show the Pythagorean theorem, using area, for a triangle with legs of length 40 units and 9 units and a hypotenuse of 41 units. Have students draw a diagram to accompany their explanation.

Summarize, or ask students to summarize, the main points from the lesson:

- We know the proof of the Pythagorean theorem using similarity better than before.
- We can prove the Pythagorean theorem using what we know about similar figures, generally, and what we know about similar triangles, specifically.
- We know a proof for the Pythagorean theorem that uses area.

Lesson Summary

The Pythagorean theorem can be proven by showing that the sum of the areas of the squares constructed off of the legs of a right triangle is equal to the area of the square constructed off of the hypotenuse of the right triangle.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 15: Pythagorean Theorem, Revisited

Exit Ticket

Explain a proof of the Pythagorean theorem in your own words. Use diagrams and concrete examples, as necessary, to support your explanation.

Exit Ticket Sample Solutions

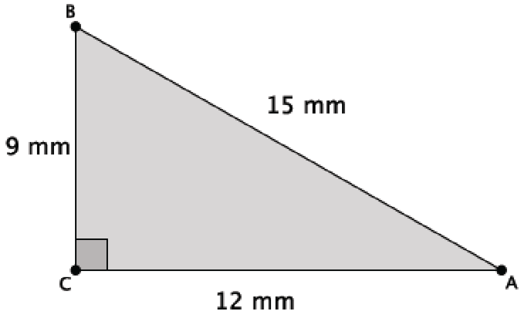
Explain a proof of the Pythagorean theorem in your own words. Use diagrams and concrete examples, as necessary, to support your explanation.

Proofs will vary. The critical parts of the proof that demonstrate proficiency include an explanation of the similar triangles $\triangle ADC$, $\triangle ACB$, *and* $\triangle CDB$, *including a statement about the ratio of their corresponding sides being equal, leading to the conclusion of the proof.*

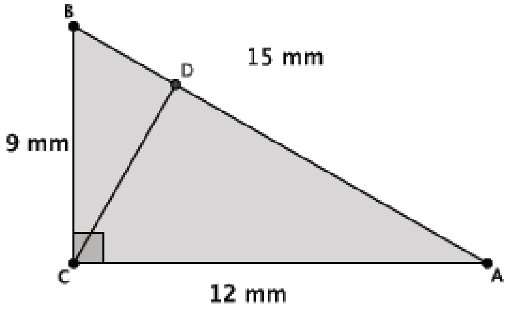
Problem Set Sample Solutions

Students apply the concept of similar figures to show the Pythagorean theorem is true.

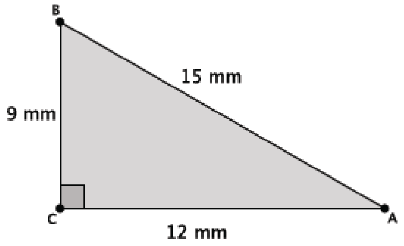
1. **For the right triangle shown below, identify and use similar triangles to illustrate the Pythagorean theorem.**

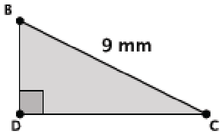


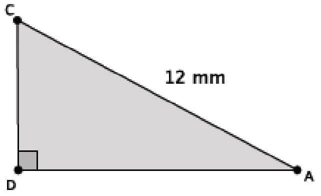
First, I must draw a segment that is perpendicular to side AB *that goes through point* C. *The point of intersection of that segment and side* AB *will be marked as point* D.



Then, I have three similar triangles, $\triangle ABC$, $\triangle CBD$, *and* $\triangle ACD$, *as shown below.*







$\triangle ABC$ and $\triangle CBD$ are similar because each one has a right angle, and they both share $\angle B$. By AA criterion, $\triangle ABC \sim \triangle CBD$. $\triangle ABC$ and $\triangle ACD$ are similar because each one has a right angle, and they both share $\angle A$. By AA criterion, $\triangle ABC \sim \triangle ACD$. By the transitive property, we also know that $\triangle ACD \sim \triangle CBD$.

Since the triangles are similar, they have corresponding sides that are equal in ratio. $\triangle ABC$ and $\triangle CBD$,

$$\frac{9}{15} = \frac{|BD|}{9},$$

which is the same as $9^2 = 15(|BD|)$.

For $\triangle ABC$ and $\triangle ACD$,

$$\frac{12}{15} = \frac{|AD|}{12},$$

which is the same as $12^2 = 15(|AD|)$.

Adding these two equations together I get

$$9^2 + 12^2 = 15(|BD|) + 15(|AD|).$$

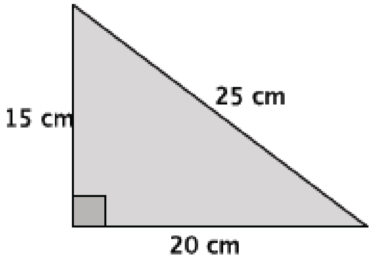
By the distributive property,

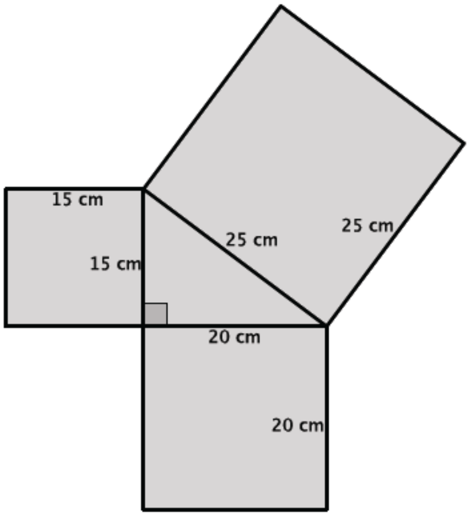
$$9^2 + 12^2 = 15(|BD| + |AD|);$$

however, $|BD| + |AD| = |AC| = 15$. Therefore,

$$9^2 + 12^2 = 15(15)$$
$$9^2 + 12^2 = 15^2.$$

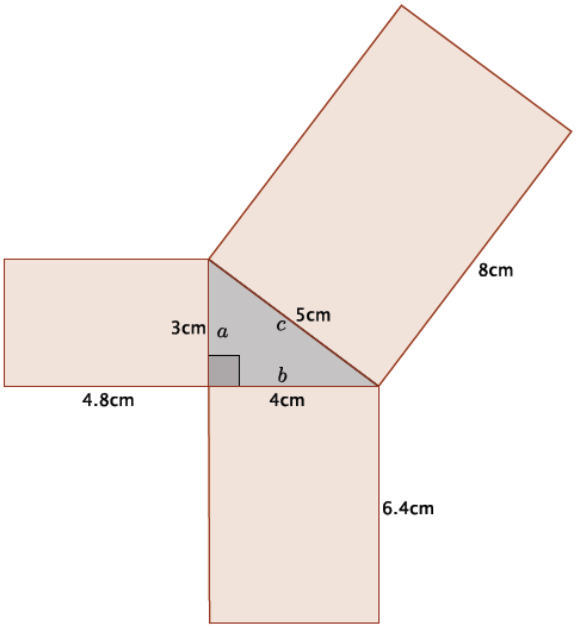
2. **For the right triangle shown below, identify and use squares formed by the sides of the triangle to illustrate the Pythagorean theorem.**





The sum of the areas of the smallest squares is $15^2 + 20^2 = 625\text{ cm}^2$. The area of the largest square is $25^2 = 625\text{ cm}^2$. The sum of the areas of the squares off of the legs is equal to the area of the square off of the hypotenuse; therefore, $a^2 + b^2 = c^2$.

3. **Reese claimed that any figure can be drawn off the sides of a right triangle and that as long as they are similar figures, then the sum of the areas off of the legs will equal the area off of the hypotenuse. She drew the diagram by constructing rectangles off of each side of a known right triangle. Is Reese's claim correct for this example? In order to prove or disprove Reese's claim, you must first show that the rectangles are similar. If they are, then you can use computations to show that the sum of the areas of the figures off of the sides a and b equals the area of the figure off of side c.**



The rectangles are similar because their corresponding side lengths are equal in scale factor. That is, if we compare the longest side of the rectangle to the side with the same length as the right triangle sides, we get the ratios

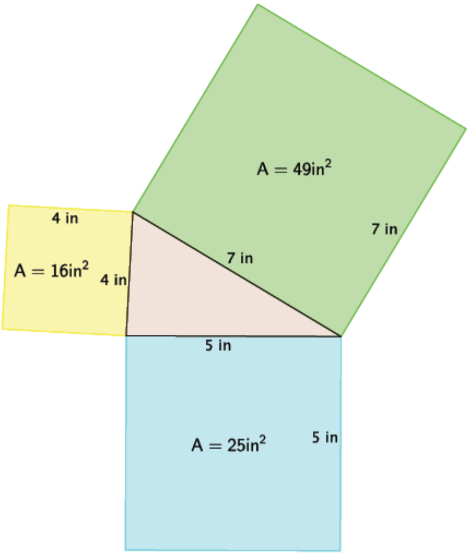
$$\frac{4.8}{3} = \frac{6.4}{4} = \frac{8}{5} = 1.6$$

Since the corresponding sides were all equal to the same constant, then we know we have similar rectangles. The areas of the smaller rectangles are 14.4 cm^2 and 25.6 cm^2, and the area of the largest rectangle is 40 cm^2. The sum of the smaller areas is equal to the larger area:

$$14.4 + 25.6 = 40$$
$$40 = 40$$

Therefore, we have shown that the sum of the areas of the two smaller rectangles is equal to the area of the larger rectangle, and Reese's claim is correct.

4. **After learning the proof of the Pythagorean theorem using areas of squares, Joseph got really excited and tried explaining it to his younger brother. He realized during his explanation that he had done something wrong. Help Joseph find his error. Explain what he did wrong.**



Based on the proof shown in class, we would expect the sum of the two smaller areas to be equal to the sum of the larger area, i.e., $16 + 25$ should equal 49. However, $16 + 25 = 41$. Joseph correctly calculated the areas of each square, so that was not his mistake. His mistake was claiming that a triangle with sides lengths of 4, 5, and 7 was a right triangle. We know that the Pythagorean theorem only works for right triangles. Considering the converse of the Pythagorean theorem, when we use the given side lengths, we do not get a true statement.

$$4^2 + 5^2 = 7^2$$
$$16 + 25 = 49$$
$$41 \neq 49$$

Therefore, the triangle Joseph began with is not a right triangle, so it makes sense that the areas of the squares were not adding up like we expected.

5. **Draw a right triangle with squares constructed off of each side that Joseph can use the next time he wants to show his younger brother the proof of the Pythagorean theorem.**

   Answers will vary. Verify that students begin, in fact, with a right triangle and do not make the same mistake that Joseph did. Consider having students share their drawings and explanations of the proof in future class meetings.

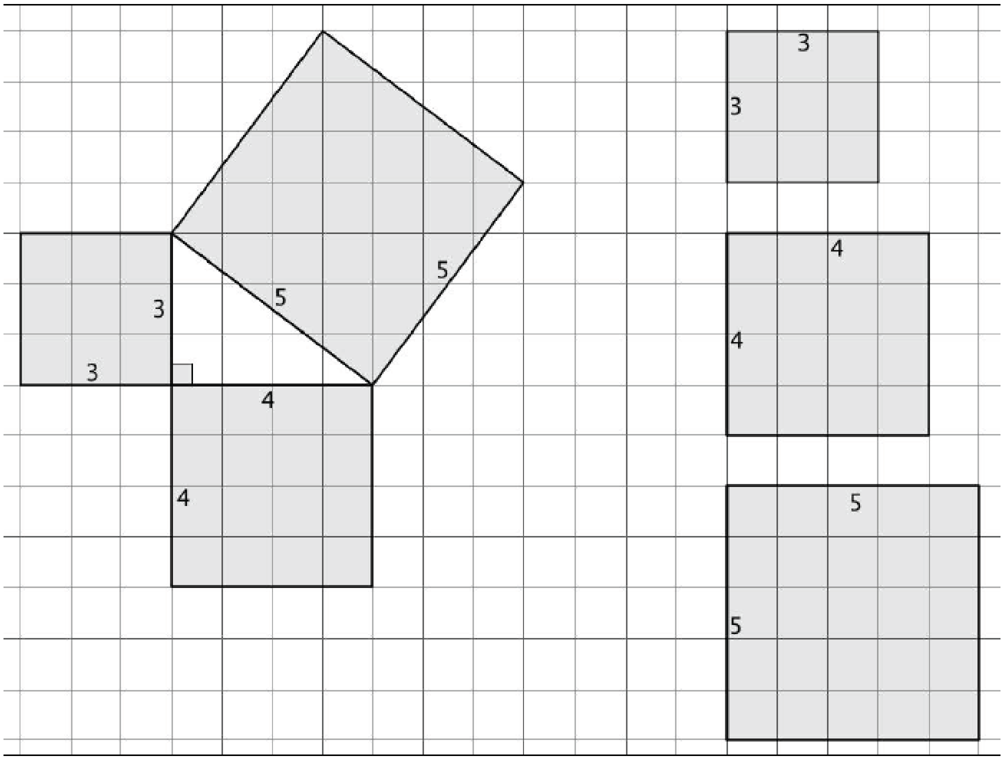
6. **Explain the meaning of the Pythagorean theorem in your own words.**

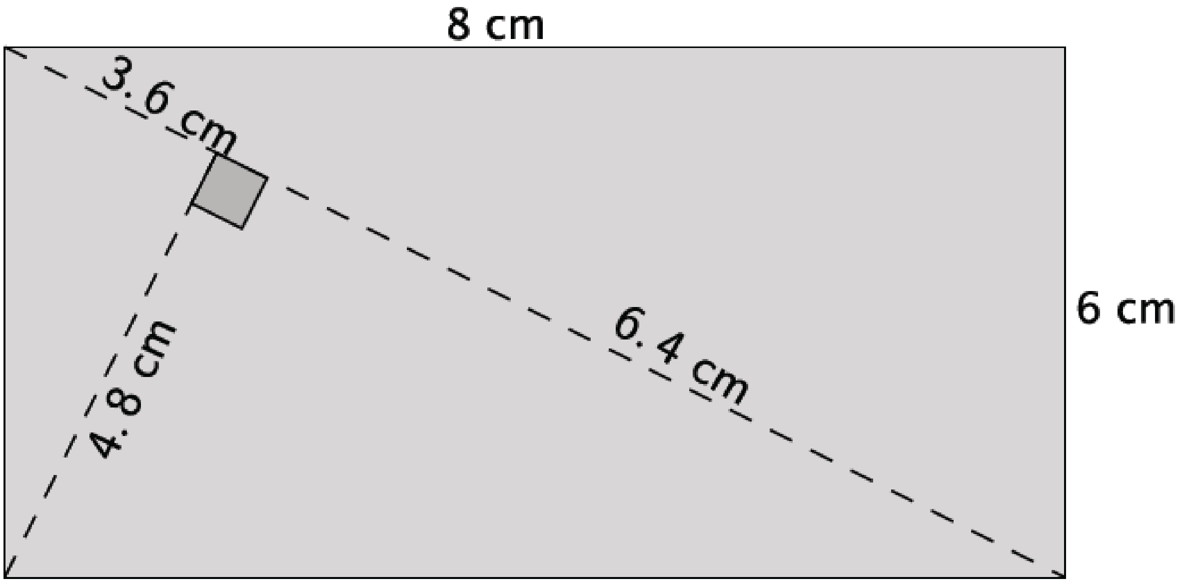
   If a triangle is a right triangle, then the sum of the squares of the legs will be equal to the square of the hypotenuse. Specifically, if the leg lengths are a and b, and the hypotenuse is length c, then for right triangles $a^2 + b^2 = c^2$.

7. **Draw a diagram that shows an example illustrating the Pythagorean theorem.**

   Diagrams will vary. Verify that students draw a right triangle with side lengths that satisfy the Pythagorean theorem.

Diagrams referenced in scaffolding boxes can be reproduced for student use.





Lesson 16: Converse of the Pythagorean Theorem

Student Outcomes

- Students explain a proof of the converse of the Pythagorean theorem.
- Students apply the theorem and its converse to solve problems.

Lesson Notes

Students had their first experience with the converse of the Pythagorean theorem in Module 3, Lesson 14. In that lesson, students learned the proof of the converse by contradiction. That is, students were asked to draw a triangle with sides a, b, c, where the angle between side a and b is greater than $90°$. The proof using the Pythagorean theorem led students to an expression that was not possible; that is, two times a length was equal to zero. This contradiction meant that the angle between sides a and b was in fact $90°$. In this lesson, students are given two triangles with base and height dimensions of a and b. They are told that one of the triangles is a right triangle and has lengths that satisfy the Pythagorean theorem. Students must use computation and their understanding of the basic rigid motions to show that the triangle with an unmarked angle is also a right triangle. The proof is subtle, so it is important from the beginning that students understand the differences between the triangles used in the discussion of the proof of the converse.

Classwork

Discussion (20 minutes)

- So far you have seen three different proofs of the Pythagorean theorem:

  THEOREM: *If the lengths of the legs of a right triangle are a and b, and the length of the hypotenuse is c, then* $a^2 + b^2 = c^2$.

MP.3

Provide students time to explain to a partner a proof of the Pythagorean theorem. Allow them to choose any one of the three proofs they have seen. Remind them of the proof from Module 2 that was based on congruent triangles, knowledge about angle sum of a triangle, and angles on a line. Also remind them of the proof from Module 3 that was based on their knowledge of similar triangles and corresponding sides being equal in ratio. Select students to share their proofs with the class. Encourage other students to critique the reasoning of the student providing the proof.

- What do you recall about the meaning of the word converse?

Consider pointing out the hypothesis and conclusion of the Pythagorean theorem and then asking students to describe the converse in those terms.

  - *The converse is when the hypothesis and conclusion of a theorem are reversed.*

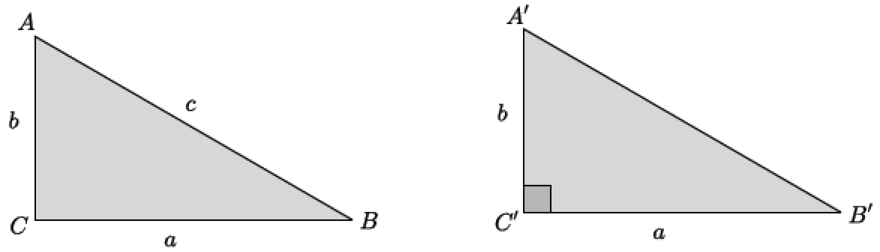
Scaffolding:

Provide students samples of converses (and note that converses are not always true):

- If it is a right angle, then the angle measure is $90°$. Converse: If the angle measure is $90°$, then it is a right angle.
- If it is raining, I will study inside the house. Converse: If I study inside the house, it is raining.

- You have also seen one proof of the converse:
    - *If the lengths of three sides of a triangle a, b, and c satisfy $c^2 = a^2 + b^2$, then the triangle is a right triangle, and furthermore, the side of length c is opposite the right angle.*
- The following is another proof of the converse. Assume we are given a triangle ABC so that the sides a, b, and c satisfy $c^2 = a^2 + b^2$. We want to show that $\angle ACB$ is a right angle. To do so, we construct a right triangle $A'B'C'$ with leg lengths of a and b and right angle $\angle A'C'B'$.

Proof of the Converse of the Pythagorean Theorem



- What do we know or not know about each of these triangles?
    - *In the first triangle, ABC, we know that $a^2 + b^2 = c^2$. We do not know if angle C is a right angle. In the second triangle, $A'B'C'$, we know that it is a right triangle.*
- What conclusions can we draw from this?
    - *By applying the Pythagorean theorem to $\triangle A'B'C'$, we get $|A'B'|^2 = a^2 + b^2$. Since we are given $c^2 = a^2 + b^2$, then by substitution, $|A'B'|^2 = c^2$, and then $|A'B'| = c$. Since c is also $|AB|$, then $|A'B'| = |AB|$. That means that both triangles have sides a, b, and c that are the exact same lengths. Therefore, if we translated one triangle along a vector (or applied any required rigid motion(s)), it would map onto the other triangle showing a congruence. Any congruence preserves the degree of angles. Which means that $\angle ACB$ is a right angle, or $90° = \angle A'C'B' = \angle ACB$.*

MP.3 Provide students time to explain to a partner a proof of the converse of the Pythagorean theorem. Allow them to choose either proof that they have seen. Remind them of the proof from Module 3 that was a proof by contradiction, where we assumed that the triangle was not a right triangle and then showed that the assumption was wrong. Select students to share their proofs with the class. Encourage other students to critique the reasoning of the student providing the proof.

Exercises 1–7 (15 minutes)

Students complete Exercises 1–7 independently. Remind students that since each of the exercises references the side length of a triangle, we need only consider the positive square root of each number, because we cannot have a negative length.

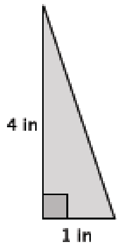
Exercises 1–7

1. **Is the triangle with leg lengths of 3 mi., 8 mi., and hypotenuse of length $\sqrt{73}$ mi. a right triangle? Show your work, and answer in a complete sentence.**

$$3^2 + 8^2 = \left(\sqrt{73}\right)^2$$
$$9 + 64 = 73$$
$$73 = 73$$

Yes, the triangle with leg lengths of 3 mi., 8 mi., and hypotenuse of length $\sqrt{73}$ mi. is a right triangle because it satisfies the Pythagorean theorem.

2. **What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.**

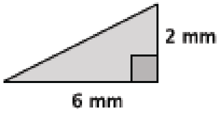


Let c represent the hypotenuse of the triangle.

$$1^2 + 4^2 = c^2$$
$$1 + 16 = c^2$$
$$17 = c^2$$
$$\sqrt{17} = c$$
$$4.1 \approx c$$

The length of the hypotenuse of the right triangle is exactly $\sqrt{17}$ inches and approximately 4.1 inches.

3. **What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.**



Let c represent the hypotenuse of the triangle.

$$2^2 + 6^2 = c^2$$
$$4 + 36 = c^2$$
$$40 = c^2$$
$$\sqrt{40} = c$$
$$\sqrt{2^3} \times \sqrt{5} = c$$
$$\sqrt{2^2} \times \sqrt{2} \times \sqrt{5} = c$$
$$2\sqrt{10} = c$$

The length of the hypotenuse of the right triangle is exactly $2\sqrt{10}$ mm and approximately 6.3 mm.

4. **Is the triangle with leg lengths of 9 in., 9 in., and hypotenuse of length $\sqrt{175}$ in. a right triangle? Show your work, and answer in a complete sentence.**

$$9^2 + 9^2 = \left(\sqrt{175}\right)^2$$
$$81 + 81 = 175$$
$$162 \neq 175$$

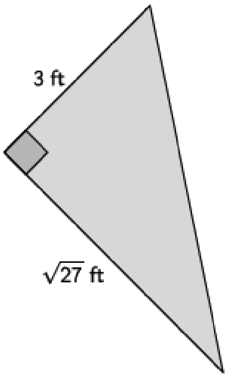
No, the triangle with leg lengths of 9 in., 9 in., *and hypotenuse of length* $\sqrt{175}$ in. *is not a right triangle because the lengths do not satisfy the Pythagorean theorem.*

5. **Is the triangle with leg lengths of $\sqrt{28}$ cm, 6 cm, and hypotenuse of length 8 cm a right triangle? Show your work, and answer in a complete sentence.**

$$\left(\sqrt{28}\right)^2 + 6^2 = 8^2$$
$$28 + 36 = 64$$
$$64 = 64$$

Yes, the triangle with leg lengths of $\sqrt{28}$ cm, 6 cm, *and hypotenuse of length* 8 cm *is a right triangle because the lengths satisfy the Pythagorean theorem.*

6. **What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence.**

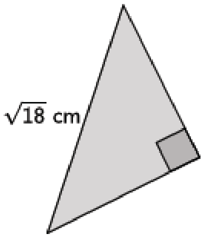


Let c *represent the hypotenuse of the triangle.*

$$3^2 + \left(\sqrt{27}\right)^2 = c^2$$
$$9 + 27 = c^2$$
$$36 = c^2$$
$$\sqrt{36} = \sqrt{c^2}$$
$$6 = c$$

The length of the hypotenuse of the right triangle is 6 ft.

7. **The triangle shown below is an isosceles right triangle. Determine the length of the legs of the triangle. Show your work, and answer in a complete sentence.**



Let x *represent the length of the side of the isosceles triangle.*

$$x^2 + x^2 = \left(\sqrt{18}\right)^2$$
$$2x^2 = 18$$
$$\frac{2x^2}{2} = \frac{18}{2}$$
$$x^2 = 9$$
$$\sqrt{x^2} = \sqrt{9}$$
$$x = 3$$

The leg lengths of the isosceles triangle are 3 cm.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- The converse of the Pythagorean theorem states that if side lengths of a triangle a, b, c satisfy $a^2 + b^2 = c^2$, then the triangle is a right triangle.
- If the side lengths of a triangle a, b, c do not satisfy $a^2 + b^2 = c^2$, then the triangle is not a right triangle.
- We know how to explain a proof of the Pythagorean theorem and its converse.

Lesson Summary

The converse of the Pythagorean theorem states that if a triangle with side lengths a, b, and c satisfies $a^2 + b^2 = c^2$, then the triangle is a right triangle.

The converse can be proven using concepts related to congruence.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 16: The Converse of the Pythagorean Theorem

Exit Ticket

1. Is the triangle with leg lengths of 7 mm, 7 mm, and a hypotenuse of length 10 mm a right triangle? Show your work, and answer in a complete sentence.

2. What would the length of hypotenuse need to be so that the triangle in Problem 1 would be a right triangle? Show work that leads to your answer.

3. If one of the leg lengths is 7 mm, what would the other leg length need to be so that the triangle in Problem 1 would be a right triangle? Show work that leads to your answer.

Exit Ticket Sample Solutions

1. **Is the triangle with leg lengths of 7 mm, 7 mm, and a hypotenuse of length 10 mm a right triangle? Show your work, and answer in a complete sentence.**

$$7^2 + 7^2 = 10^2$$
$$49 + 49 = 100$$
$$98 \neq 100$$

No, the triangle with leg lengths of 7 mm, 7 mm, and hypotenuse of length 10 mm is not a right triangle because the lengths do not satisfy the Pythagorean theorem.

2. **What would the length of the hypotenuse need to be so that the triangle in Problem 1 would be a right triangle? Show work that leads to your answer.**

Let c represent the length of the hypotenuse in millimeters.

Then,

$$7^2 + 7^2 = c^2$$
$$49 + 49 = c^2$$
$$98 = c^2$$
$$\sqrt{98} = c$$

The hypotenuse would need to be $\sqrt{98} \text{ mm}$ for the triangle with sides of 7 mm and 7 mm to be a right triangle.

3. **If one of the leg lengths is 7 mm, what would the other leg length need to be so that the triangle in Problem 1 would be a right triangle? Show work that leads to your answer.**

Let a represent the length of one leg in millimeters.

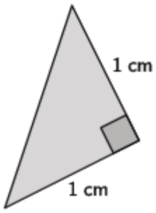
Then,

$$a^2 + 7^2 = 10^2$$
$$a^2 + 49 = 100$$
$$a^2 + 49 - 49 = 100 - 49$$
$$a^2 = 51$$
$$a = \sqrt{51}$$

The leg length would need to be $\sqrt{51} \text{ mm}$ so that the triangle with one leg length of 7 mm and the hypotenuse of 10 mm is a right triangle.

Problem Set Sample Solutions

1. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.

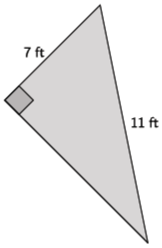


Let c represent the length of the hypotenuse of the triangle in centimeters.

$$1^2 + 1^2 = c^2$$
$$1 + 1 = c^2$$
$$2 = c^2$$
$$\sqrt{2} = \sqrt{c^2}$$
$$1.4 \approx c$$

The length of the hypotenuse is exactly $\sqrt{2}$ cm *and approximately* 1.4 cm.

2. What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.



Let x represent the unknown length of the triangle in feet.

$$7^2 + x^2 = 11^2$$
$$49 + x^2 = 121$$
$$49 - 49 + x^2 = 121 - 49$$
$$x^2 = 72$$
$$\sqrt{x^2} = \sqrt{72}$$
$$x = \sqrt{2^2} \times \sqrt{2} \times \sqrt{3^2}$$
$$x = 6\sqrt{2}$$
$$x \approx 8.5$$

The length of the unknown side of the triangle is exactly $6\sqrt{2}$ ft. *and approximately* 8.5 ft.

3. Is the triangle with leg lengths of $\sqrt{3}$ cm, 9 cm, and hypotenuse of length $\sqrt{84}$ cm a right triangle? Show your work, and answer in a complete sentence.

$$\left(\sqrt{3}\right)^2 + 9^2 = \left(\sqrt{84}\right)^2$$
$$3 + 81 = 84$$
$$84 = 84$$

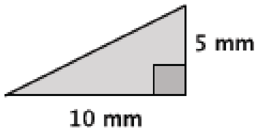
Yes, the triangle with leg lengths of $\sqrt{3}$ cm, 9 cm, *and hypotenuse of length* $\sqrt{84}$ cm *is a right triangle because the lengths satisfy the Pythagorean theorem.*

4. Is the triangle with leg lengths of $\sqrt{7}$ km, 5 km, and hypotenuse of length $\sqrt{48}$ km a right triangle? Show your work, and answer in a complete sentence.

$$\left(\sqrt{7}\right)^2 + 5^2 = \left(\sqrt{48}\right)^2$$
$$7 + 25 = 48$$
$$32 \neq 48$$

No, the triangle with leg lengths of $\sqrt{7}$ km, 5 km, *and hypotenuse of length* $\sqrt{48}$ km *is not a right triangle because the lengths do not satisfy the Pythagorean theorem.*

5. **What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.**



Let c represent the length of the hypotenuse of the triangle in millimeters.

$$\begin{aligned} 5^2+10^2&=c^2\\ 25+100&=c^2\\ 125&=c^2\\ \sqrt{125}&=\sqrt{c^2}\\ \sqrt{5^3}&=c\\ \sqrt{5^2}\times\sqrt{5}&=c\\ 5\sqrt{5}&=c\\ 11.2&\approx c \end{aligned}$$

The length of the hypotenuse is exactly $5\sqrt{5}$ mm *and approximately* 11.2 mm.

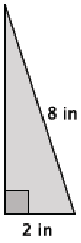
6. **Is the triangle with leg lengths of 3, 6, and hypotenuse of length $\sqrt{45}$ a right triangle? Show your work, and answer in a complete sentence.**

$$\begin{aligned} 3^2+6^2&=\left(\sqrt{45}\right)^2\\ 9+36&=45\\ 45&=45 \end{aligned}$$

Yes, the triangle with leg lengths of 3 and 6 and hypotenuse of length $\sqrt{45}$ is a right triangle because the lengths satisfy the Pythagorean theorem.

7. **What is the length of the unknown side of the right triangle shown below? Show your work, and answer in a complete sentence. Provide an exact answer and an approximate answer rounded to the tenths place.**

Let x represent the unknown side length of the triangle in inches.



$$\begin{aligned} 2^2+x^2&=8^2\\ 4+x^2&=64\\ 4-4+x^2&=64-4\\ x^2&=60\\ \sqrt{x^2}&=\sqrt{60}\\ x&=\sqrt{2^2}\times\sqrt{3}\times\sqrt{5}\\ x&=2\sqrt{15}\\ x&\approx 7.7 \end{aligned}$$

The length of the unknown side of the triangle is exactly $2\sqrt{15}$ inches and approximately 7.7 inches.

8. **Is the triangle with leg lengths of 1 and $\sqrt{3}$ and hypotenuse of length 2 a right triangle? Show your work, and answer in a complete sentence.**

$$\begin{aligned} 1^2+\left(\sqrt{3}\right)^2&=2^2\\ 1+3&=4\\ 4&=4 \end{aligned}$$

Yes, the triangle with leg lengths of 1 and $\sqrt{3}$ and hypotenuse of length 2 is a right triangle because the lengths satisfy the Pythagorean theorem.

9. Corey found the hypotenuse of a right triangle with leg lengths of 2 and 3 to be $\sqrt{13}$. Corey claims that since $\sqrt{13} = 3.61$ when estimating to two decimal digits, that a triangle with leg lengths of 2 and 3 and a hypotenuse of 3.61 is a right triangle. Is he correct? Explain.

No, Corey is not correct.

$$2^2 + 3^2 = (3.61)^2$$
$$4 + 9 = 13.0321$$
$$13 \neq 13.0321$$

No, the triangle with leg lengths of 2 and 3 and hypotenuse of length 3.61 is not a right triangle because the lengths do not satisfy the Pythagorean theorem.

10. Explain a proof of the Pythagorean theorem.

Consider having students share their proof with a partner while their partner critiques their reasoning. Accept any of the three proofs that the students have seen.

11. Explain a proof of the converse of the Pythagorean theorem.

Consider having students share their proof with a partner while their partner critiques their reasoning. Accept either of the proofs that the students have seen.

Lesson 17: Distance on the Coordinate Plane

Student Outcomes

- Students determine the distance between two points on a coordinate plane using the Pythagorean theorem.

Lesson Notes

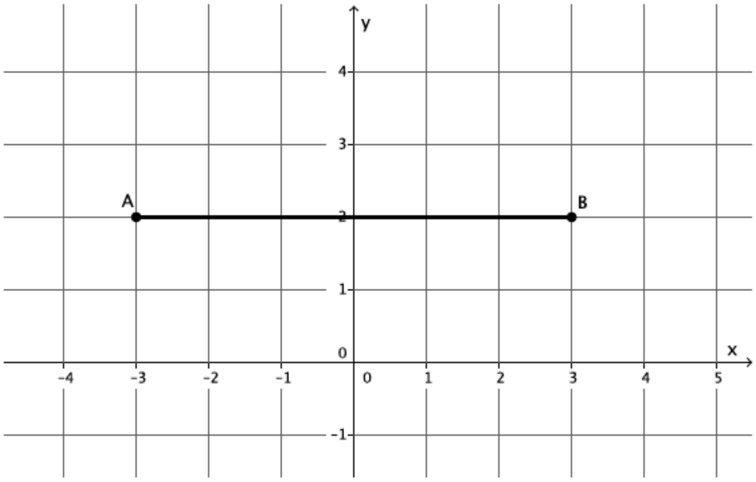
Calculators will be helpful in this lesson for determining values of radical expressions.

Classwork

Example 1 (6 minutes)

Example 1

What is the distance between the two points A and B on the coordinate plane?

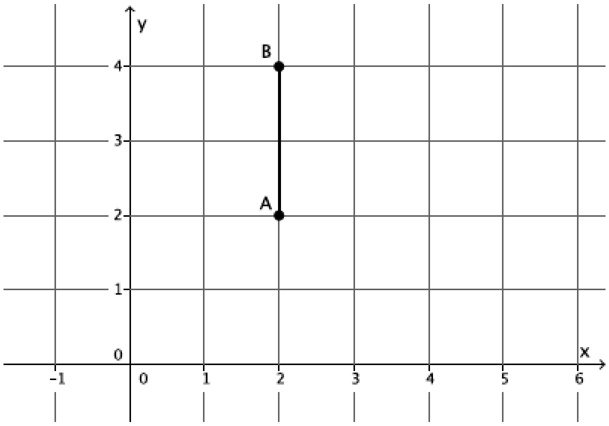


- What is the distance between the two points A and B on the coordinate plane?
    - *The distance between points A and B is 6 units.*

Scaffolding:

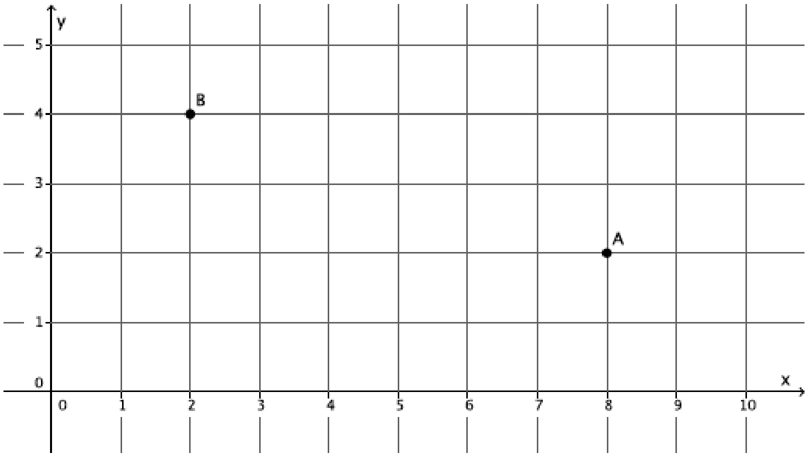
Students may benefit from physically measuring lengths to understand finding distance. A reproducible of cut-outs for this example has been included at the end of the lesson.

What is the distance between the two points A and B on the coordinate plane?



- What is the distance between the two points A and B on the coordinate plane?
  - *The distance between points A and B is 2 units.*

What is the distance between the two points A and B on the coordinate plane? Round your answer to the tenths place.

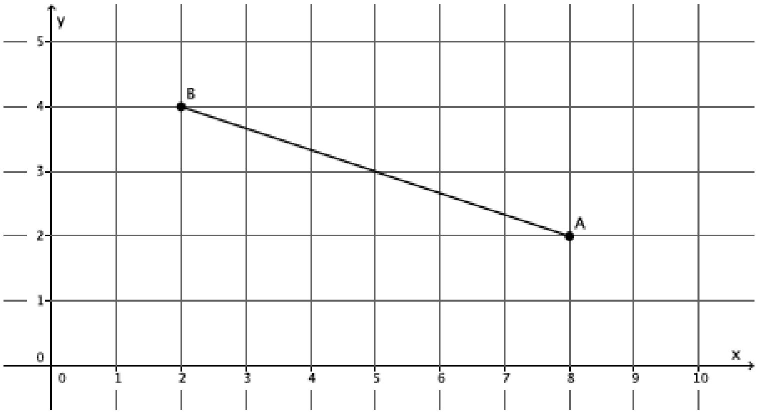


- What is the distance between the two points A and B on the coordinate plane? Round your answer to the tenths place.

Provide students time to solve the problem. Have students share their work and estimations of the distance between the points. The questions below can be used to guide students' thinking.

MP.1 & MP.7

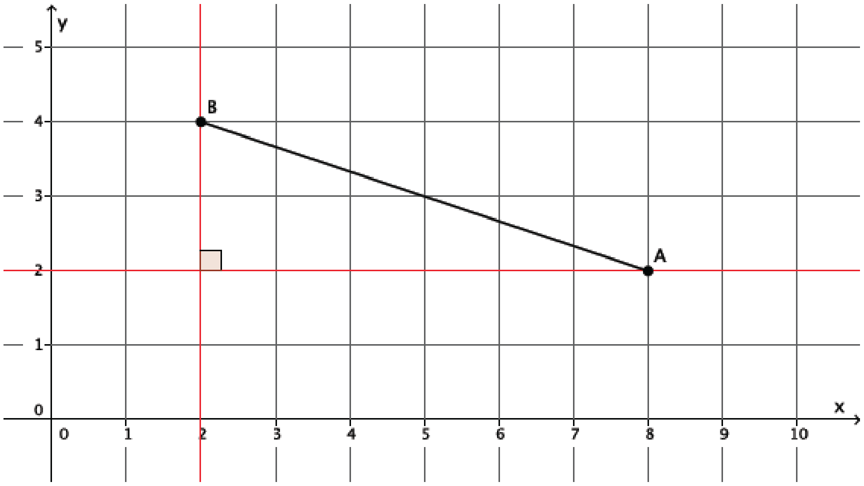
- We cannot simply count units between the points because the line that connects A to B is not horizontal or vertical. What have we done recently that allowed us to find the length of an unknown segment?



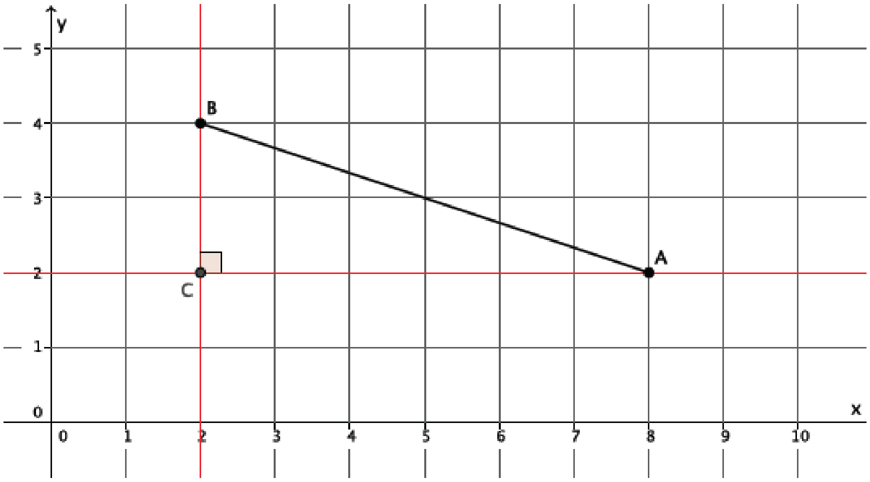
  - *The Pythagorean theorem allows us to determine the length of an unknown side of a right triangle.*
- Use what you know about the Pythagorean theorem to determine the distance between points A and B.

Provide students time to solve the problem now that they know that the Pythagorean theorem can help them. If necessary, the questions below can guide students' thinking.

- We must draw a right triangle so that $|AB|$ is the hypotenuse. How can we construct the right triangle that we need?
  - *Draw a vertical line through* B *and a horizontal line through* A*. Or, draw a vertical line through* A *and a horizontal line through* B.



- Let's mark the point of intersection of the horizontal and vertical lines we drew as point C. What is the length of $|AC|$? $|BC|$?



  - *The length of* $|AC| = 6$ *units, and the length of* $|BC| = 2$ *units.*

- Now that we know the lengths of the legs of the right triangle, we can determine the length of $|AB|$.

Remind students that because we are finding a length, we need only consider the positive value of the square root because a negative length does not make sense. If necessary, remind students of this fact throughout their work in this lesson.

  - *Let* c *be the length of* AB.

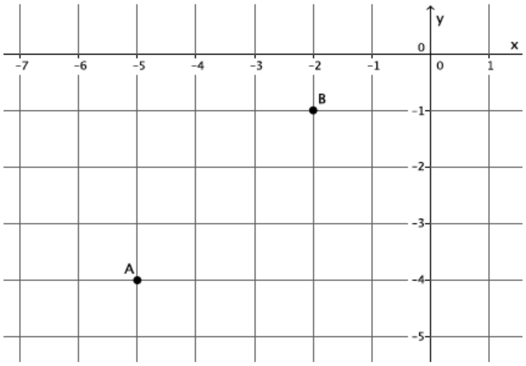
$$\begin{aligned} 2^2 + 6^2 &= c^2 \\ 4 + 36 &= c^2 \\ 40 &= c^2 \\ \sqrt{40} &= c \\ 6.3 &\approx c \end{aligned}$$

The distance between points A *and* B *is approximately* 6.3 *units.*

Example 2 (6 minutes)

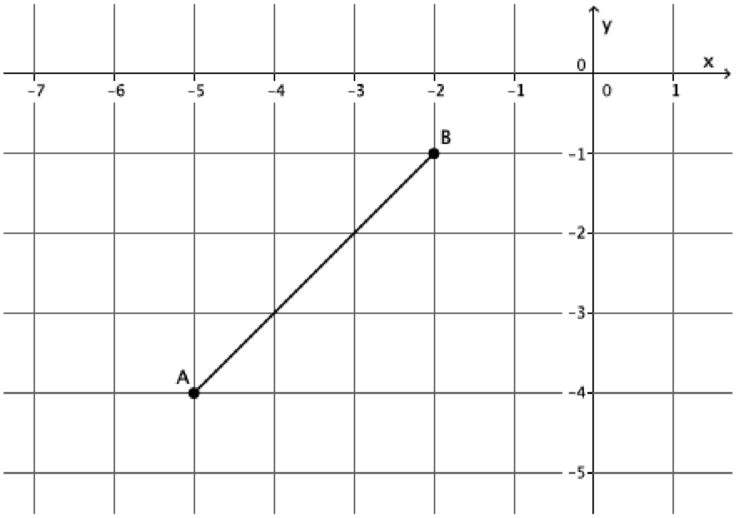
Example 2

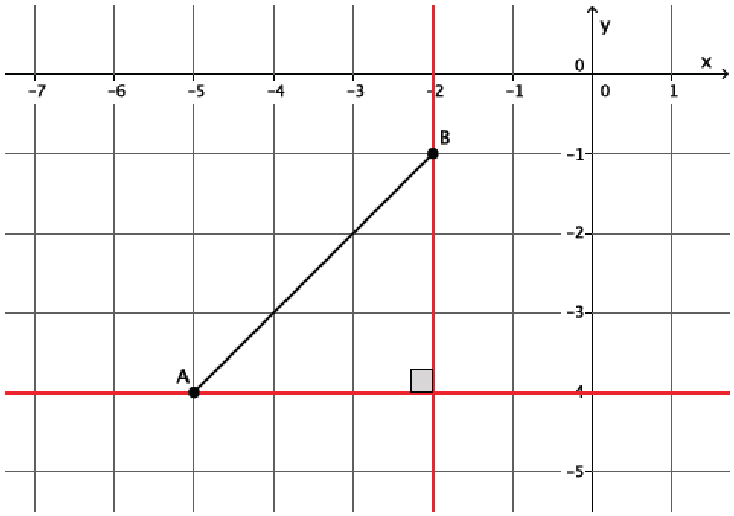
Given two points A and B on the coordinate plane, determine the distance between them. First, make an estimate; then, try to find a more precise answer. Round your answer to the tenths place.



Provide students time to solve the problem. Have students share their work and estimations of the distance between the points. The questions below can be used to guide students' thinking.

- We know that we need a right triangle. How can we draw one?

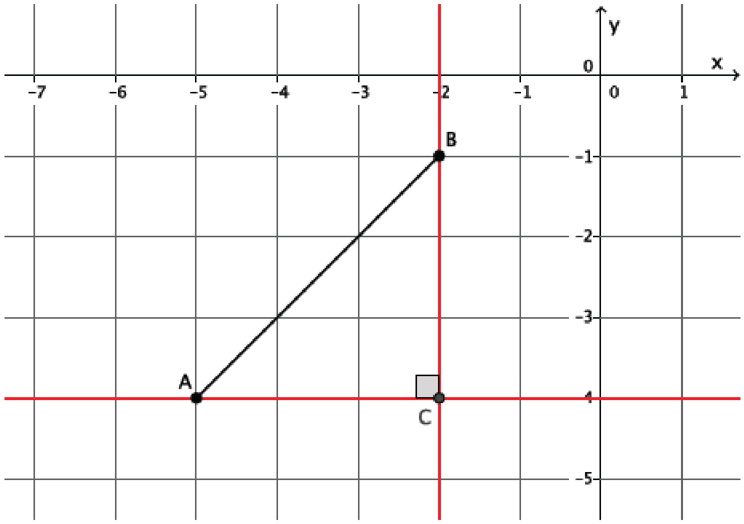




- *Draw a vertical line through B and a horizontal line through A. Or draw a vertical line through A and a horizontal line through B.*

- Mark the point C at the intersection of the horizontal and vertical lines. What do we do next?
  - *Count units to determine the lengths of the legs of the right triangle, and then use the Pythagorean theorem to find $|AB|$.*

Show the last diagram, and ask a student to explain the answer.



- *The length $|AC| = 3$ units, and the length $|BC| = 3$ units. Let c be $|AB|$.*

$$3^2 + 3^2 = c^2$$
$$9 + 9 = c^2$$
$$18 = c^2$$
$$\sqrt{18} = c$$
$$4.2 \approx c$$

The distance between points A and B is approximately 4.2 units.

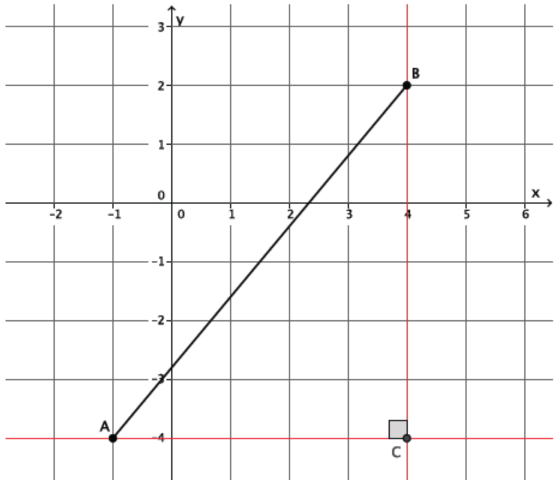
Exercises 1–4 (12 minutes)

Students complete Exercises 1–4 independently.

Exercises 1–4

For each of the Exercises 1–4, determine the distance between points A and B on the coordinate plane. Round your answer to the tenths place.

1.

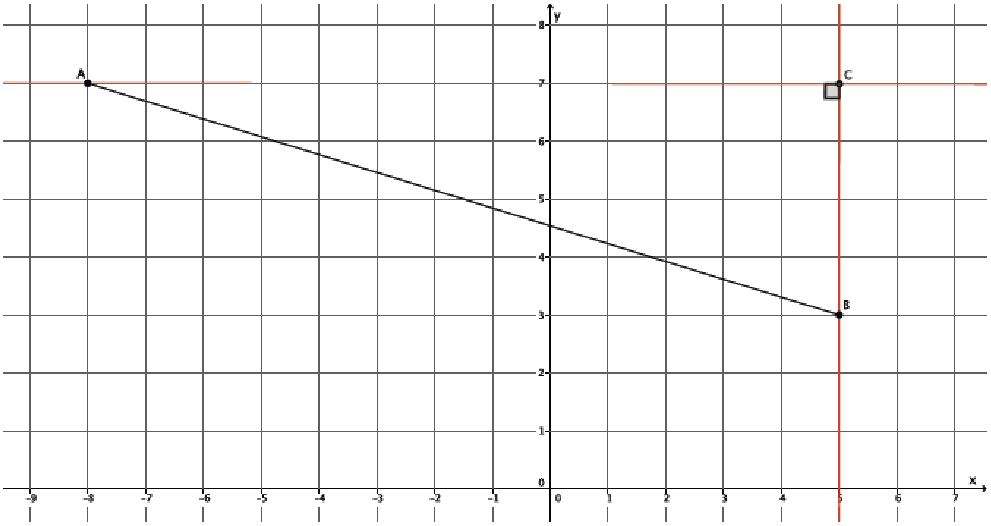


Let c represent $|AB|$.

$$5^2 + 6^2 = c^2$$
$$25 + 36 = c^2$$
$$61 = c^2$$
$$\sqrt{61} = c$$
$$7.8 \approx c$$

The distance between points A and B is about 7.8 units.

2.

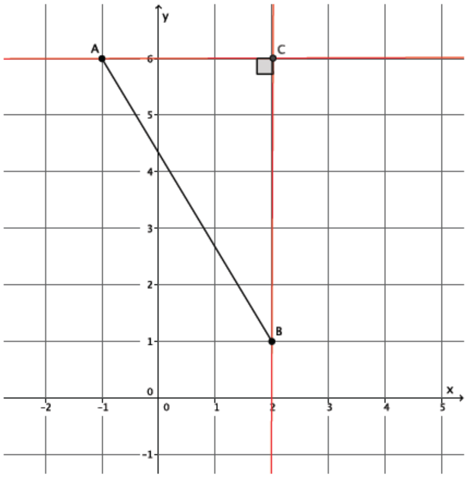


Let c represent $|AB|$.

$$13^2 + 4^2 = c^2$$
$$169 + 16 = c^2$$
$$185 = c^2$$
$$\sqrt{185} = c$$
$$13.6 \approx c$$

The distance between points A and B is about 13.6 units.

3.

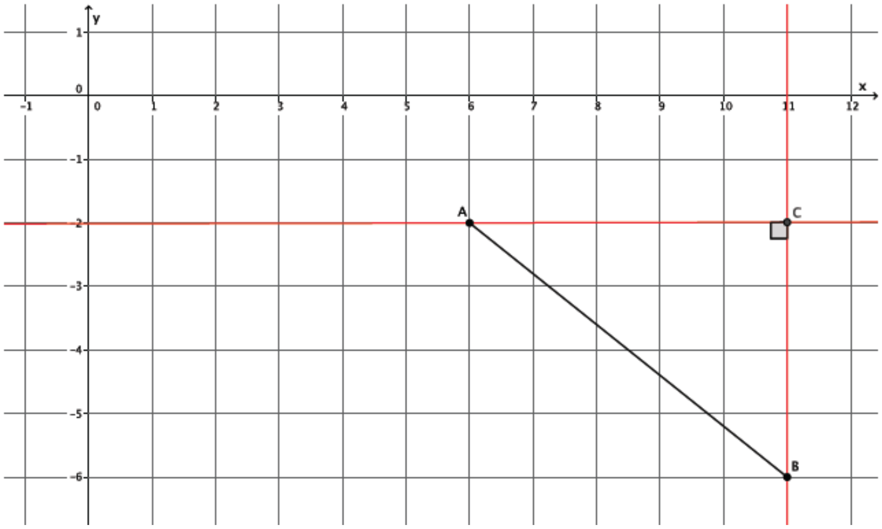


Let c represent $|AB|$.

$$3^2 + 5^2 = c^2$$
$$9 + 25 = c^2$$
$$34 = c^2$$
$$\sqrt{34} = c$$
$$5.8 \approx c$$

The distance between points A and B is about 5.8 units.

4.



Let c represent $|AB|$.

$$5^2 + 4^2 = c^2$$
$$25 + 16 = c^2$$
$$41 = c^2$$
$$\sqrt{41} = c$$
$$6.4 \approx c$$

The distance between points A and B is about 6.4 units.

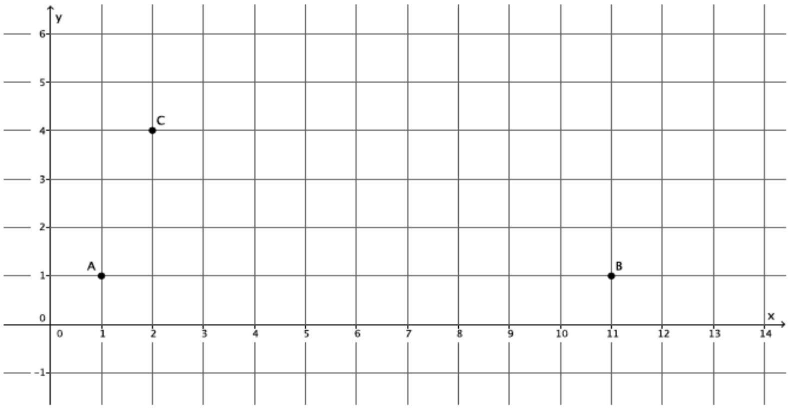
Example 3 (14 minutes)

- Is the triangle formed by the points A, B, C a right triangle?

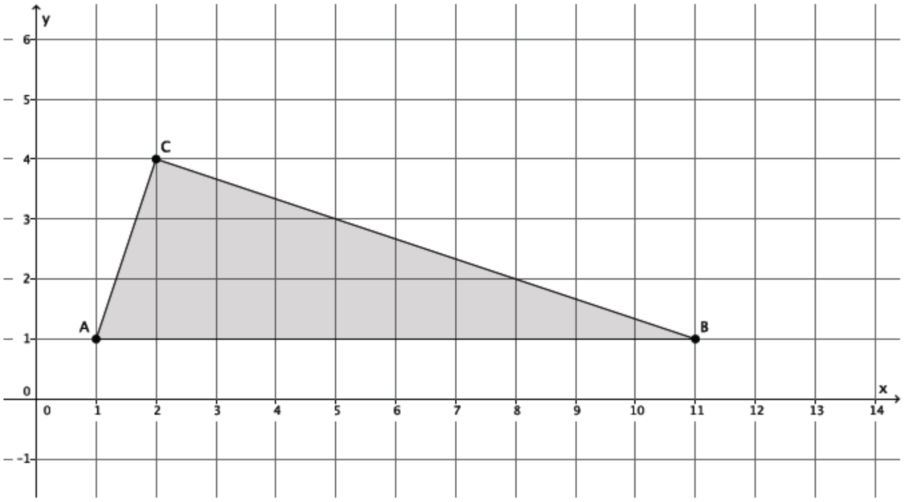
Provide time for small groups of students to discuss and determine if the triangle formed is a right triangle. Have students share their reasoning with the class. If necessary, use the questions below to guide their thinking.

Example 3

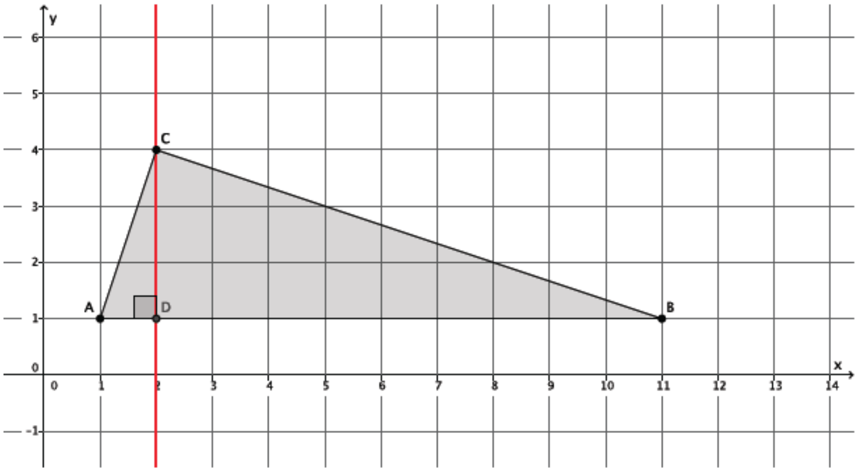
Is the triangle formed by the points A, B, C a right triangle?



- How can we verify if a triangle is a right triangle?
  - *Use the converse of the Pythagorean theorem.*
- What information do we need about the triangle in order to use the converse of the Pythagorean theorem, and how would we use it?
  - *We need to know the lengths of all three sides; then, we can check to see if the side lengths satisfy the Pythagorean theorem.*



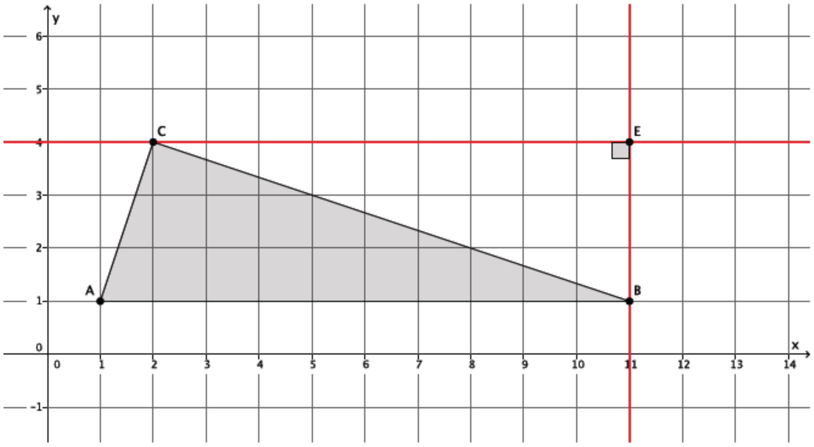
- Clearly, the length of $|AB| = 10$ units. How can we determine $|AC|$?
  - *To find $|AC|$, follow the same steps used in the previous problem. Draw horizontal and vertical lines to form a right triangle, and use the Pythagorean theorem to determine the length.*
- Determine $|AC|$. Leave your answer in square root form unless it is a perfect square.



  - *Let c represent $|AC|$.*

$$1^2 + 3^2 = c^2$$
$$1 + 9 = c^2$$
$$10 = c^2$$
$$\sqrt{10} = c$$

- Now, determine $|BC|$. Again, leave your answer in square root form unless it is a perfect square.



  - *Let c represent $|BC|$.*

$$9^2 + 3^2 = c^2$$
$$81 + 9 = c^2$$
$$90 = c^2$$
$$\sqrt{90} = c$$

- The lengths of the three sides of the triangle are 10 units, $\sqrt{10}$ units, and $\sqrt{90}$ units. Which number represents the hypotenuse of the triangle? Explain.
    - *The side AB must be the hypotenuse because it is the longest side. When estimating the lengths of the other two sides, I know that $\sqrt{10}$ is between 3 and 4, and $\sqrt{90}$ is between 9 and 10. Therefore, the side that is 10 units in length is the hypotenuse.*
- Use the lengths 10, $\sqrt{10}$, and $\sqrt{90}$ to determine if the triangle is a right triangle.
    - *Sample Response*

$$\begin{aligned}\left(\sqrt{10}\right)^2 + \left(\sqrt{90}\right)^2 &= 10^2\\ 10 + 90 &= 100\\ 100 &= 100\end{aligned}$$

- Therefore, the points A, B, C form a right triangle.

Closing (3 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- To find the distance between two points on the coordinate plane, draw a right triangle and use the Pythagorean theorem.
- To verify if a triangle in the plane is a right triangle, use both the Pythagorean theorem and its converse.

Lesson Summary

To determine the distance between two points on the coordinate plane, begin by connecting the two points. Then, draw a vertical line through one of the points and a horizontal line through the other point. The intersection of the vertical and horizontal lines forms a right triangle to which the Pythagorean theorem can be applied.

To verify if a triangle is a right triangle, use the converse of the Pythagorean theorem.

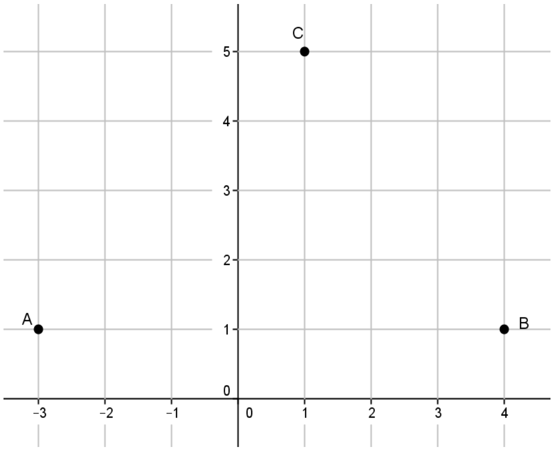
Exit Ticket (4 minutes)

Name ______________________________ Date ______________

Lesson 17: Distance on the Coordinate Plane

Exit Ticket

Use the following diagram to answer the questions below.



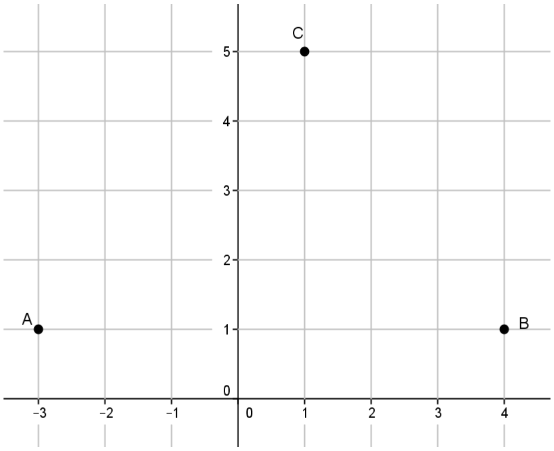
1. Determine $|AC|$. Leave your answer in square root form unless it is a perfect square.

2. Determine $|CB|$. Leave your answer in square root form unless it is a perfect square.

3. Is the triangle formed by the points A, B, C a right triangle? Explain why or why not.

Exit Ticket Sample Solutions

Use the following diagram to answer the questions below.



1. **Determine $|AC|$. Leave your answer in square root form unless it is a perfect square.**

   Let c represent $|AC|$.

$$\begin{aligned} 4^2 + 4^2 &= c^2 \\ 16 + 16 &= c^2 \\ 32 &= c^2 \\ \sqrt{32} &= c \end{aligned}$$

2. **Determine $|CB|$. Leave your answer in square root form unless it is a perfect square.**

   Let d represent $|CB|$.

$$\begin{aligned} 3^2 + 4^2 &= d^2 \\ 9 + 16 &= d^2 \\ 25 &= d^2 \\ \sqrt{25} &= d \\ 5 &= d \end{aligned}$$

3. **Is the triangle formed by the points A, B, C a right triangle? Explain why or why not.**

   Using the lengths $5, \sqrt{32}$, and $|AB| = 7$ to determine if the triangle is a right triangle, I have to check to see if

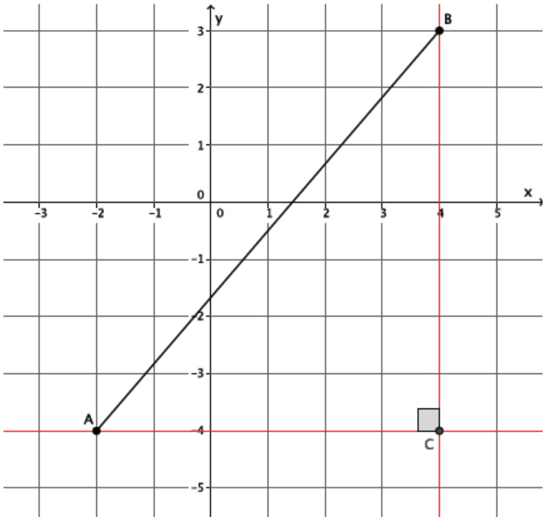
$$\begin{aligned} 5^2 + (\sqrt{32})^2 &= 7^2 \\ 25 + 32 &\neq 49 \end{aligned}$$

   Therefore, the triangle formed by the points A, B, C is not a right triangle because the lengths of the triangle do not satisfy the Pythagorean theorem.

Problem Set Sample Solutions

For each of the Problems 1–4, determine the distance between points A and B on the coordinate plane. Round your answer to the tenths place.

1.

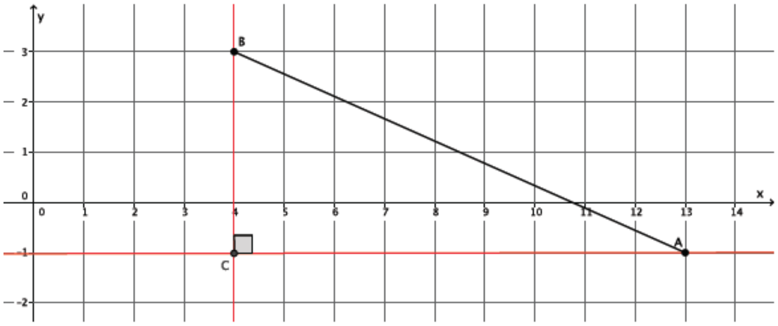


Let c represent $|AB|$.

$$6^2 + 7^2 = c^2$$
$$36 + 49 = c^2$$
$$85 = c^2$$
$$\sqrt{85} = c$$
$$9.2 \approx c$$

The distance between points A and B is about 9.2 units.

2.

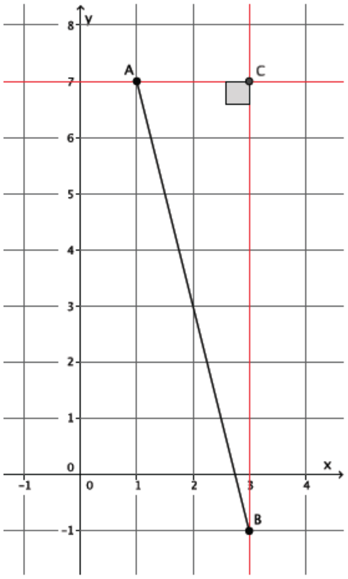


Let c represent $|AB|$.

$$9^2 + 4^2 = c^2$$
$$81 + 16 = c^2$$
$$97 = c^2$$
$$\sqrt{97} = c$$
$$9.8 \approx c$$

The distance between points A and B is about 9.8 units.

3.

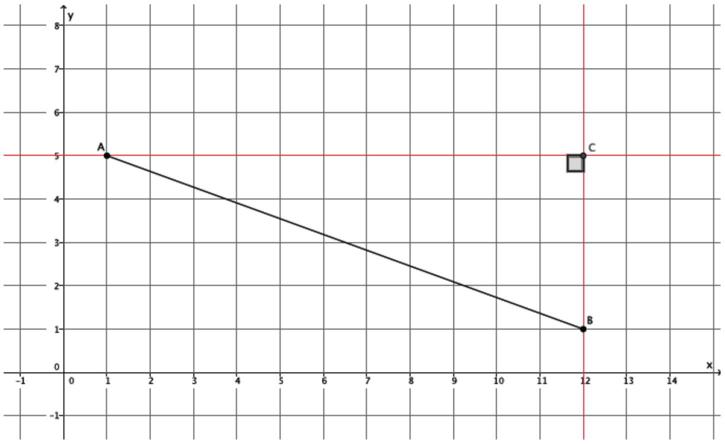


Let c represent $|AB|$.

$$2^2 + 8^2 = c^2$$
$$4 + 64 = c^2$$
$$68 = c^2$$
$$\sqrt{68} = c$$
$$8.2 \approx c$$

The distance between points A and B is about 8.2 units.

4.

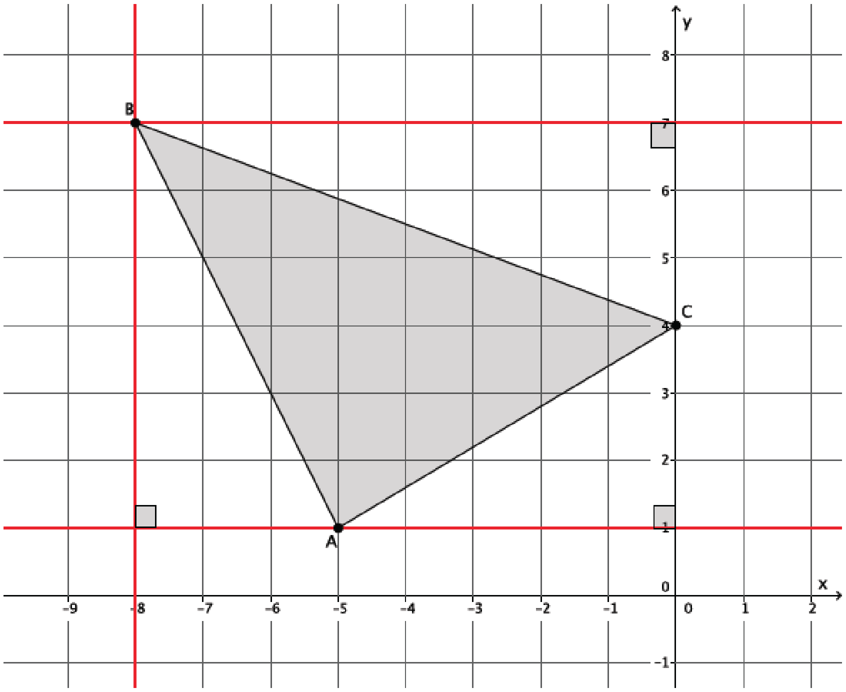


Let c represent $|AB|$.

$$11^2 + 4^2 = c^2$$
$$121 + 16 = c^2$$
$$137 = c^2$$
$$\sqrt{137} = c$$
$$11.7 \approx c$$

The distance between points A and B is about 11.7 units.

5. **Is the triangle formed by points A, B, C a right triangle?**



Let c represent $|AB|$.

$$3^2 + 6^2 = c^2$$
$$9 + 36 = c^2$$
$$45 = c^2$$
$$\sqrt{45} = c$$

Let c represent $|AC|$.

$$3^2 + 5^2 = c^2$$
$$9 + 25 = c^2$$
$$34 = c^2$$
$$\sqrt{34} = c$$

Let c represent $|BC|$.

$$3^2 + 8^2 = c^2$$
$$9 + 64 = c^2$$
$$73 = c^2$$
$$\sqrt{73} = c$$

$$\left(\sqrt{45}\right)^2 + \left(\sqrt{34}\right)^2 = \left(\sqrt{73}\right)^2$$
$$45 + 34 = 73$$
$$79 \neq 73$$

No, the points do not form a right triangle.

Lesson 18: Applications of the Pythagorean Theorem

Student Outcomes

- Students apply the Pythagorean theorem to real-world and mathematical problems in two dimensions.

Lesson Notes

It is recommended that students have access to a calculator as they work through the exercises. However, it is not recommended that students use calculators to answer the questions but only to check their work or estimate the value of an irrational number using rational approximation. Make clear to students that they can use calculators but that all mathematical work should be shown. This lesson includes a Fluency Exercise that will take approximately 10 minutes to complete. The Fluency Exercise is a white board exchange with problems on volume that can be found at the end of this lesson. It is recommended that the Fluency Exercise take place at the beginning of the lesson or after the discussion that concludes the lesson.

Classwork

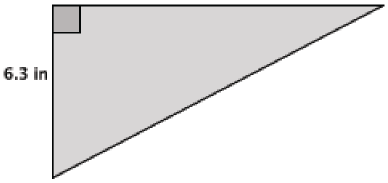
Exploratory Challenge/Exercises 1–5 (20 minutes)

MP.1

Students complete Exercises 1–5 in pairs or small groups. These problems are applications of the Pythagorean theorem and are an opportunity to remind students of Mathematical Practice 1: Make sense of problems and persevere in solving them. Students should compare their solutions and solution methods in their pairs, small groups, and as a class. If necessary, remind students that we are finding lengths, which means we need only consider the positive square root of a number.

Exercises

1. **The area of the right triangle shown below is 26.46 in^2. What is the perimeter of the right triangle? Round your answer to the tenths place.**



Let b represent the base of the triangle in inches where $h = 6.3$.

$$A = \frac{bh}{2}$$
$$26.46 = \frac{6.3b}{2}$$
$$52.92 = 6.3b$$
$$\frac{52.92}{6.3} = \frac{6.3b}{6.3}$$
$$8.4 = b$$

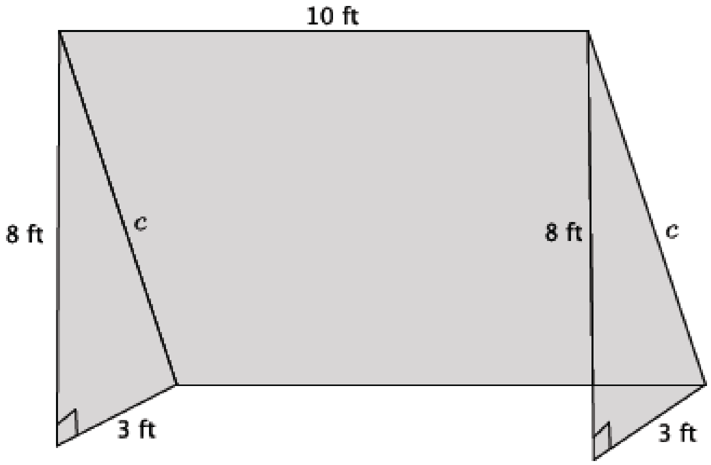
Let c represent the length of the hypotenuse in inches.

$$6.3^2 + 8.4^2 = c^2$$
$$39.69 + 70.56 = c^2$$
$$110.25 = c^2$$
$$\sqrt{110.25} = \sqrt{c^2}$$
$$\sqrt{110.25} = c$$

The number $\sqrt{110.25}$ is between 10 and 11. When comparing with tenths, the number is actually equal to 10.5 because $10.5^2 = 110.25$. Therefore, the length of the hypotenuse is 10.5 in.

The perimeter of the triangle is 6.3 in. $+ 8.4$ in. $+ 10.5$ in. $= 25.2$ in.

2. The diagram below is a representation of a soccer goal.



a. Determine the length of the bar, c, that would be needed to provide structure to the goal. Round your answer to the tenths place.

Let c represent the hypotenuse of the right triangle in feet.

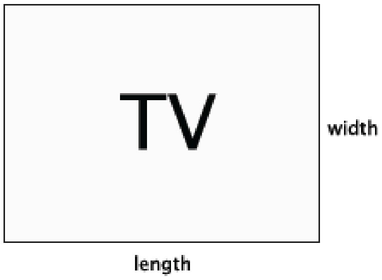
$$8^2 + 3^2 = c^2$$
$$64 + 9 = c^2$$
$$73 = c^2$$
$$\sqrt{73} = \sqrt{c^2}$$

The number $\sqrt{73}$ is between 8 and 9. In the sequence of tenths, it is between 8.5 and 8.6 because $8.5^2 < \left(\sqrt{73}\right)^2 < 8.6^2$. In the sequence of hundredths, the number is between 8.54 and 8.55 because $8.54^2 < \left(\sqrt{73}\right)^2 < 8.55^2$. Since the number $\sqrt{73}$ is between 8.54 and 8.55, it would round to 8.5. The length of the bar that provides structure for the goal is 8.5 ft.

b. How much netting (in square feet) is needed to cover the entire goal?

The area of the triangles are each 12 ft^2. The area of the rectangle in the back is 85 ft^2. The total area of netting required to cover the goal is 109 ft^2.

3. The typical ratio of length to width that is used to produce televisions is $4:3$.



Note to Teacher:

Check in with students to make sure they understand how TVs are measured in terms of their diagonal length, and what is meant by the ratio of $4:3$. To complete the problem, students must be clear that the size of a TV is not denoted by its length or width but by the length of the diagonal (hypotenuse). Also, students must have some sense that the ratio of length to width must be some multiple of the ratio $4:3$; otherwise, the TV would not give a good perspective. Consider showing students what a TV would look like with a ratio of $9:12$. They should notice that such dimensions yield a TV screen that is unfamiliar to them.

a. A TV with those exact measurements would be quite small, so generally the size of the television is enlarged by multiplying each number in the ratio by some factor of x. For example, a reasonably sized television might have dimensions of $4 \times 5 : 3 \times 5$, where the original ratio $4:3$ was enlarged by a scale factor of 5. The size of a television is described in inches, such as a 60" TV, for example. That measurement actually refers to the diagonal length of the TV (distance from an upper corner to the opposite lower corner). What measurement would be applied to a television that was produced using the ratio of $4 \times 5 : 3 \times 5$?

Let c be the length of the diagonal in inches.

$$\begin{aligned} 20^2 + 15^2 &= c^2 \\ 400 + 225 &= c^2 \\ 625 &= c^2 \\ \sqrt{625} &= \sqrt{c^2} \\ 25 &= c \end{aligned}$$

Since the TV has a diagonal length of 25 inches, then it is a 25" TV.

b. A 42" TV was just given to your family. What are the length and width measurements of the TV?

Let x be the factor applied to the ratio $4:3$.

$$\begin{aligned} (3x)^2 + (4x)^2 &= 42^2 \\ 9x^2 + 16x^2 &= 1764 \\ (9 + 16)x^2 &= 1764 \\ 25x^2 &= 1764 \\ \frac{25x^2}{25} &= \frac{1764}{25} \\ x^2 &= 70.56 \\ \sqrt{x^2} &= \sqrt{70.56} \\ x &= \sqrt{70.56} \end{aligned}$$

The number $\sqrt{70.56}$ is between 8 and 9. In working with the sequence of tenths, I realized the number $\sqrt{70.56}$ is actually equal to 8.4 because $8.4^2 = 70.56$. Therefore, $x = 8.4$, and the dimensions of the TV are $4 \times 8.4 = 33.6$ inches and $3 \times 8.4 = 25.2$ inches.

c. Check that the dimensions you got in part (b) are correct using the Pythagorean theorem.

$$\begin{aligned} 33.6^2 + 25.2^2 &= 42^2 \\ 1128.96 + 635.04 &= 1764 \\ 1764 &= 1764 \end{aligned}$$

d. The table that your TV currently rests on is 30" in length. Will the new TV fit on the table? Explain.

The dimension for the length of the TV is 33.6 inches. It will not fit on a table that is 30 inches in length.

4. **Determine the distance between the following pairs of points. Round your answer to the tenths place. Use graph paper if necessary.**

   a. $(7, 4)$ and $(-3, -2)$

   Let c represent the distance between the two points.

$$10^2 + 6^2 = c^2$$
$$100 + 36 = c^2$$
$$136 = c^2$$
$$\sqrt{136} = \sqrt{c^2}$$
$$\sqrt{136} = c$$

   The number $\sqrt{136}$ is between 11 and 12. In the sequence of tenths, it is between 11.6 and 11.7 because $11.6^2 < \left(\sqrt{136}\right)^2 < 11.7^2$. In the sequence of hundredths, it is between 11.66 and 11.67, which means the number will round to 11.6. The distance between the two points is 11.6 units.

   b. $(-5, 2)$ and $(3, 6)$

   Let c represent the distance between the two points.

$$8^2 + 4^2 = c^2$$
$$64 + 16 = c^2$$
$$80 = c^2$$
$$\sqrt{80} = \sqrt{c^2}$$
$$\sqrt{80} = c$$

   The number $\sqrt{80}$ is between 8 and 9. In the sequence of tenths, it is between 8.9 and 9 because $8.9^2 < \left(\sqrt{80}\right)^2 < 9^2$. In the sequence of hundredths, it is between 8.94 and 8.95, which means it will round to 8.9. The distance between the two points is 8.9 units.

   c. Challenge: (x_1, y_1) and (x_2, y_2). Explain your answer.

   Note: Deriving the distance formula using the Pythagorean theorem is not part of the standard but does present an interesting challenge to students. Assign it only to students who need a challenge.

   Let c represent the distance between the two points.

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = c^2$$
$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{c^2}$$
$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = c$$

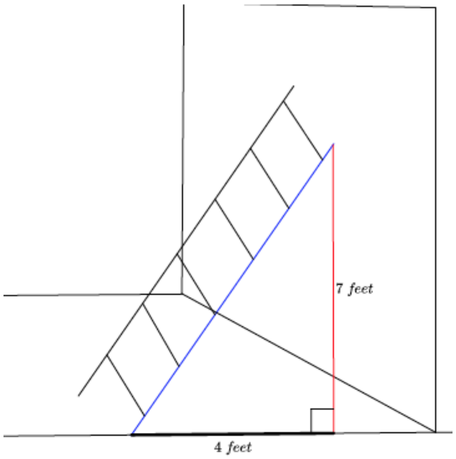
   I noticed that the dimensions of the right triangle were equal to the difference in x-values and difference in y-values. Using those expressions and what I knew about solving radical equations, I was able to determine the length of c.

5. **What length of ladder will be needed to reach a height of 7 feet along the wall when the base of the ladder is 4 feet from the wall? Round your answer to the tenths place.**

Let c represent the length of the ladder in feet.

$$7^2 + 4^2 = c^2$$
$$49 + 16 = c^2$$
$$65 = c^2$$
$$\sqrt{65} = \sqrt{c^2}$$
$$\sqrt{65} = c$$

The number $\sqrt{65}$ is between 8 and 9. In the sequence of tenths, it is between 8 and 8.1 because $8^2 < \left(\sqrt{65}\right)^2 < 8.1^2$. In the sequence of hundredths, it is between 8.06 and 8.07, which means the number will round to 8.1. The ladder must be 8.1 feet long to reach 7 feet up a wall when placed 4 feet from the wall.



Discussion (5 minutes)

This discussion provides a challenge question to students about how the Pythagorean theorem might be applied to a three-dimensional situation. The next lesson focuses on using the Pythagorean theorem to answer questions about cones and spheres.

- The majority of our work with Pythagorean theorem has been in two dimensions. Can you think of any applications we have seen so far that are in three dimensions?
  - *The soccer goal is three-dimensional. A ladder propped up against a wall is three-dimensional.*
- What new applications of Pythagorean theorem in three dimensions do you think we will work on next?

Provide students time to think about this in pairs or small groups.

  - *We have worked with solids this year, so there may be an application involving cones and spheres.*

Fluency Exercise (10 minutes): Area and Volume II

RWBE: Refer to the Rapid White Board Exchanges section in the Module Overview for directions to administer a Rapid White Board Exchange.

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know some basic applications of the Pythagorean theorem in terms of measures of a television, length of a ladder, area and perimeter of right triangles, etc.
- We know that there will be some three-dimensional applications of the theorem beyond what we have already seen.

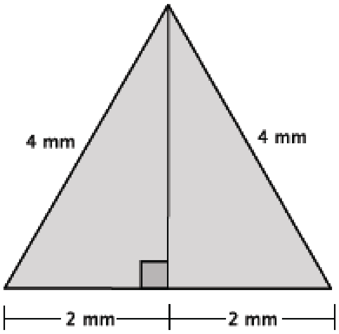
Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 18: Applications of the Pythagorean Theorem

Exit Ticket

Use the diagram of the equilateral triangle shown below to answer the following questions. Show work that leads to your answers.



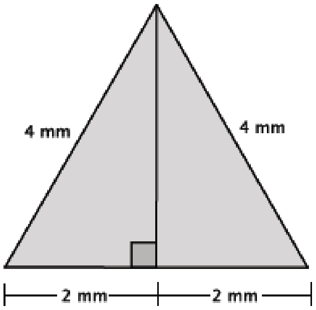
a. What is the perimeter of the triangle?

b. What is the height, h, of the equilateral triangle? Write an exact answer using a square root and approximate answer rounded to the tenths place.

c. Using the approximate height found in part (b), estimate the area of the equilateral triangle.

Exit Ticket Sample Solutions

Use the diagram of the equilateral triangle shown below to answer the following questions. Show work that leads to your answers.



a. What is the perimeter of the triangle?

$4 + 4 + 4 = 12$

The perimeter is 12 mm.

b. What is the height, h, of the equilateral triangle? Write an exact answer using a square root and approximate answer rounded to the tenths place.

Using the fact that the height is one leg length of a right triangle, and I know the hypotenuse is 4 mm *and the other leg length is* 2 mm, *I can use the Pythagorean theorem to find* h.

$$2^2 + h^2 = 4^2$$
$$4 + h^2 = 16$$
$$4 - 4 + h^2 = 16 - 4$$
$$h^2 = 12$$
$$h = \sqrt{12}$$
$$h = \sqrt{4 \times 3}$$
$$h = \sqrt{4} \times \sqrt{3}$$
$$h = 2\sqrt{3}$$

The number $\sqrt{3}$ *is between* 1 *and* 2. *In the sequence of tenths, it is between* 1.7 *and* 1.8 *because* $1.7^2 < \left(\sqrt{3}\right)^2 < 1.8^2$. *In the sequence of hundredths, it is between* 1.73 *and* 1.74, *which means it would round to* 1.7. *Then* $2 \times 1.7 = 3.4$ mm *is the approximate length of the hypotenuse, and* $\sqrt{12} = 2\sqrt{3}$ mm *is the exact length.*

c. Using the approximate height found in part (b), estimate the area of the equilateral triangle.

$$A = \frac{bh}{2}$$
$$A = \frac{4(3.4)}{2}$$
$$A = \frac{13.6}{2}$$
$$A = 6.8$$

The approximate area of the equilateral triangle is 6.8 mm^2.

Problem Set Sample Solutions

Students continue applying the Pythagorean theorem to solve real-world and mathematical problems.

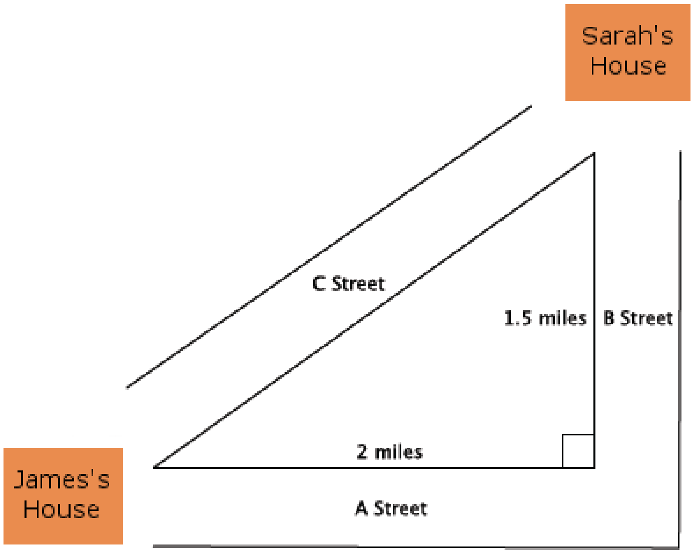
1. **A 70" TV is advertised on sale at a local store. What are the length and width of the television?**

The TV is in the ratio of $4:3$ *and has measurements of* $4x:3x$*, where* x *is the scale factor of enlargement.*

$$(3x)^2 + (4x)^2 = 70^2$$
$$9x^2 + 16x^2 = 4900$$
$$25x^2 = 4900$$
$$\frac{25x^2}{25} = \frac{4900}{25}$$
$$x^2 = 196$$
$$\sqrt{x^2} = \sqrt{196}$$
$$x = 14$$

The length of the TV is $4 \times 14 = 56$ *inches, and the width is* $3 \times 14 = 42$ *inches.*

2. **There are two paths that one can use to go from Sarah's house to James's house. One way is to take C Street, and the other way requires you to use A Street and B Street. How much shorter is the direct path along C Street?**

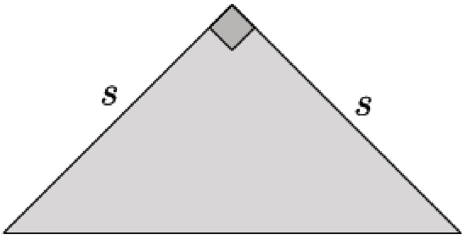


Let c *represent the length of the hypotenuse of the right triangle in miles.*

$$2^2 + 1.5^2 = c^2$$
$$4 + 2.25 = c^2$$
$$6.25 = c^2$$
$$\sqrt{6.25} = \sqrt{c^2}$$
$$2.5 = c$$

The path using A Street and B Street is 3.5 *miles. The path along C Street is* 2.5 *miles. The path along C Street is exactly* 1 *mile shorter than the path along A Street and B Street.*

3. An isosceles right triangle refers to a right triangle with equal leg lengths, s, as shown below.



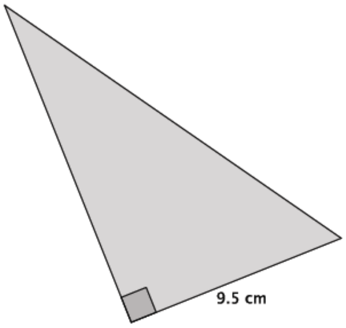
What is the length of the hypotenuse of an isosceles right triangle with a leg length of 9 cm? Write an exact answer using a square root and an approximate answer rounded to the tenths place.

Let c be the length of the hypotenuse of the isosceles triangle in centimeters.

$$\begin{aligned} 9^2 + 9^2 &= c^2 \\ 81 + 81 &= c^2 \\ 162 &= c^2 \\ \sqrt{162} &= \sqrt{c^2} \\ \sqrt{81 \times 2} &= c \\ \sqrt{81} \times \sqrt{2} &= c \\ 9\sqrt{2} &= c \end{aligned}$$

The number $\sqrt{2}$ is between 1 and 2. In the sequence of tenths, it is between 1.4 and 1.5 because $1.4^2 < \left(\sqrt{2}\right)^2 < 1.5^2$. Since the number 2 is closer to 1.4^2 than 1.5^2, it would round to 1.4. Then $9 \times 1.4 = 12.6$ cm is the approximate length of the hypotenuse, and $9\sqrt{2}$ cm is the exact length.

4. The area of the right triangle shown below is $66.5\ \text{cm}^2$.



a. What is the height of the triangle?

$$\begin{aligned} A &= \frac{bh}{2} \\ 66.5 &= \frac{9.5h}{2} \\ 133 &= 9.5h \\ \frac{133}{9.5} &= \frac{9.5h}{9.5} \\ 14 &= h \end{aligned}$$

The height of the triangle is 14 cm.

b. **What is the perimeter of the right triangle? Round your answer to the tenths place.**

Let c represent the length of the hypotenuse in centimeters.

$$9.5^2 + 14^2 = c^2$$
$$90.25 + 196 = c^2$$
$$286.25 = c^2$$
$$\sqrt{286.25} = \sqrt{c^2}$$
$$\sqrt{286.25} = c$$

The number $\sqrt{286.25}$ is between 16 and 17. In the sequence of tenths, the number is between 16.9 and 17 because $16.9^2 < \left(\sqrt{286.25}\right)^2 < 17^2$. Since 286.25 is closer to 16.9^2 than 17^2, then the approximate length of the hypotenuse is 16.9 cm.

The perimeter of the triangle is $9.5 \text{ cm} + 14 \text{ cm} + 16.9 \text{ cm} = 40.4 \text{ cm}$.

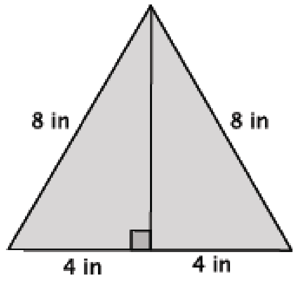
5. **What is the distance between points $(1, 9)$ and $(-4, -1)$? Round your answer to the tenths place.**

Let c represent the distance between the points.

$$10^2 + 5^2 = c^2$$
$$100 + 25 = c^2$$
$$125 = c^2$$
$$\sqrt{125} = \sqrt{c^2}$$
$$\sqrt{125} = c$$
$$11.2 \approx c$$

The distance between the points is approximately 11.2 units.

6. **An equilateral triangle is shown below. Determine the area of the triangle. Round your answer to the tenths place.**



Let h represent the height of the triangle in inches.

$$4^2 + h^2 = 8^2$$
$$16 + h^2 = 64$$
$$h^2 = 48$$
$$\sqrt{h^2} = \sqrt{48}$$
$$h = \sqrt{48}$$
$$h \approx 6.9$$

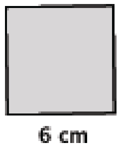
$$A = \frac{8(6.9)}{2} = 4(6.9) = 27.6$$

The area of the triangle is 27.6 in^2.

Area and Volume II

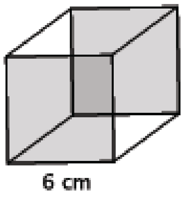
1. Find the area of the square shown below.

$A = (6\text{ cm})^2$
$= 36\text{ cm}^2$



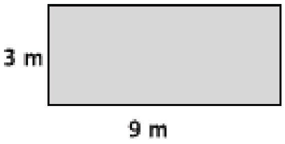
2. Find the volume of the cube shown below.

$V = (6\text{ cm})^3$
$= 216\text{ cm}^3$



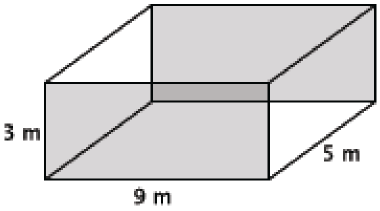
3. Find the area of the rectangle shown below.

$A = (9\text{ m})(3\text{ m})$
$= 27\text{ m}^2$



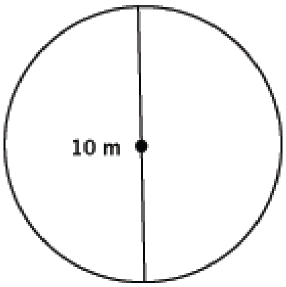
4. Find the volume of the rectangular prism show below.

$V = (27\text{ m}^2)(5\text{ m})$
$= 135\text{ m}^3$



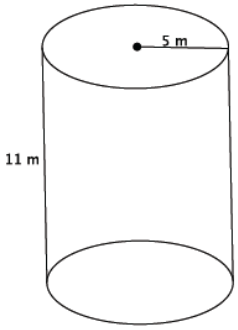
5. Find the area of the circle shown below.

$A = (5\text{ m})^2\pi$
$= 25\pi\text{ m}^2$



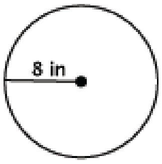
6. Find the volume of the cylinder show below.

$$V = (25\pi \text{ m}^2)(11 \text{ m})$$
$$= 275\pi \text{ m}^3$$



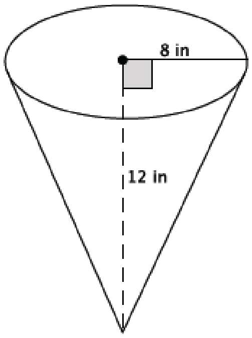
7. Find the area of the circle shown below.

$$A = (8 \text{ in.})^2\pi$$
$$= 64\pi \text{ in}^2$$



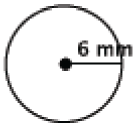
8. Find the volume of the cone show below.

$$V = \left(\frac{1}{3}\right)(64\pi \text{ in}^2)(12 \text{ in.})$$
$$= 256\pi \text{ in}^3$$



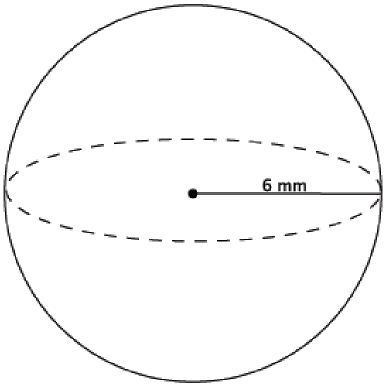
9. Find the area of the circle shown below.

$$A = (6 \text{ mm})^2\pi$$
$$= 36\pi \text{ mm}^2$$



10. Find the volume of the sphere shown below.

$$V = \left(\frac{4}{3}\right)\pi(6 \text{ mm})^3$$
$$= \frac{864 \text{ mm}}{3}\pi$$
$$= 288\pi \text{ mm}^3$$



Topic D:

Applications of Radicals and Roots

8.G.B.7, 8.G.C.9

Focus Standards:	8.G.B.7	Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.
	8.G.C.9	Know the formulas for the volumes of cones, cylinders, and spheres and use them to solve real-world and mathematical problems.
Instructional Days:	5	
Lesson 19:	Cones and Spheres (P)[1]	
Lesson 20:	Truncated Cones (P)	
Lesson 21:	Volume of Composite Solids (E)	
Lesson 22:	Average Rate of Change (S)	
Lesson 23:	Nonlinear Motion (M)	

In Lesson 19, students use the Pythagorean theorem to determine the height, lateral length (slant height), or radius of the base of a cone. Students also use the Pythagorean theorem to determine the radius of a sphere given the length of a cord. Many problems in Lesson 19 also require students to use the height, length, or radius they determined using the Pythagorean theorem to then find the volume of a figure. In Lesson 20, students learn that the volume of a truncated cone can be determined using facts about similar triangles. Specifically, the fact that corresponding parts of similar triangles are equal in ratio is used to determine the height of the part of the cone that has been removed to make the truncated cone. Then, students calculate the volume of the whole cone (i.e., removed part and truncated part) and subtract the volume of the removed portion to determine the volume of the truncated cone. In this lesson, students learn that the formula to determine the volume of a pyramid is analogous to that of a cone. That is, the volume of a pyramid is exactly one-third the volume of a rectangular prism with the same base area and height. In Lesson 21, students determine the volume of solids comprised of cylinders, cones, spheres, and combinations of those figures as composite solids. Students consistently link their understanding of expressions (numerical and algebraic) to the volumes they represent. In Lesson 22, students apply their knowledge of volume to

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[1] Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

compute the average rate of change in the height of the water level when water drains into a conical container. Students bring together much of what they have learned in Grade 8, such as Pythagorean theorem, volume of solids, similarity, constant rate, and rate of change, to work on challenging problems in Lessons 22 and 23. The optional modeling lesson, Lesson 23, challenges students with a problem about nonlinear motion. In describing the motion of a ladder sliding down a wall, students bring together concepts of exponents, roots, average speed, constant rate, functions, and the Pythagorean theorem. Throughout the lesson, students are challenged to reason abstractly and quantitatively while making sense of problems, applying their knowledge of concepts learned throughout the year to persevere in solving them.

Lesson 19: Cones and Spheres

Student Outcomes

- Students use the Pythagorean theorem to determine an unknown dimension of a cone or a sphere.
- Students know that a pyramid is a special type of cone with triangular faces and a rectangular base.
- Students know how to use the lateral length of a cone and the length of a chord of a sphere to solve problems related to volume.

Classwork

Exercises 1–2 (5 minutes)

Students complete Exercises 1–2 individually. The purpose of these exercises is for students to perform computations that may help them see how a pyramid's volume can be seen as analogous to a cone's volume. Their response to part (b) of Exercise 2 is the starting point of the discussion that follows.

Scaffolding:

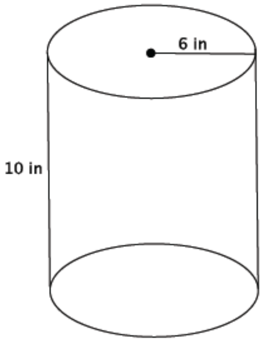
Printable nets, located at the end of the lesson, can be used to create 3-D models for Exercises 1–2.

Exercises 1–2

Note: Figures not drawn to scale.

1. **Determine the volume for each figure below.**

   a. **Write an expression that shows volume in terms of the area of the base, B, and the height of the figure. Explain the meaning of the expression, and then use it to determine the volume of the figure.**



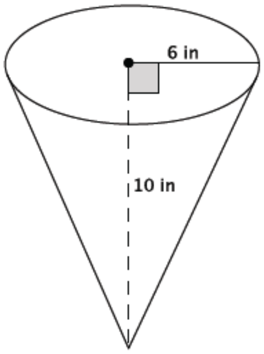
$$V = Bh$$

The expression $V = Bh$ means that the volume of the cylinder is found by multiplying the area of the base by the height. The base is a circle whose area can be found by squaring the radius, 6 in., and then multiplying by π. The volume is found by multiplying that area by the height of 10.

$$V = \pi 6^2(10)$$
$$= 360\pi$$

The volume of the cylinder is 360π in^3.

b. **Write an expression that shows volume in terms of the area of the base, B, and the height of the figure. Explain the meaning of the expression, and then use it to determine the volume of the figure.**



$$V = \frac{1}{3}Bh$$

The expression $V = \frac{1}{3}Bh$ *means that the volume of the cone is found by multiplying the area of the base by the height, then taking one-third of that product. The base is a circle whose area can be found by squaring the radius,* 6 in., *and then multiplying by* π. *The volume is found by multiplying that area by the height of* 10 in. *and then taking one-third of that product.*

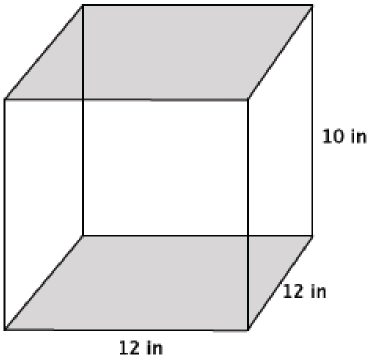
$$\begin{aligned} V &= \frac{1}{3}\pi 6^2(10) \\ &= \frac{360}{3}\pi \\ &= 120\,\pi \end{aligned}$$

The volume of the cone is 120π in^3.

2.

MP.7 & MP.8

a. **Write an expression that shows volume in terms of the area of the base, B, and the height of the figure. Explain the meaning of the expression, and then use it to determine the volume of the figure.**



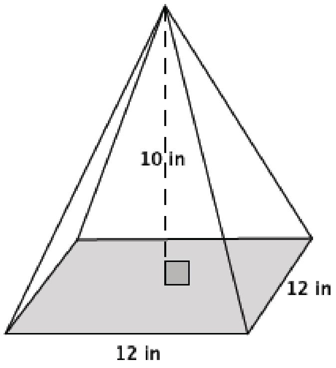
$$V = Bh$$

The expression $V = Bh$ *means that the volume of the prism is found by multiplying the area of the base by the height. The base is a square whose area can be found by multiplying* 12×12. *The volume is found by multiplying that area,* 144, *by the height of* 10.

$$\begin{aligned} V &= 12(12)(10) \\ &= 1440 \end{aligned}$$

The volume of the prism is 1440 in^3.

b. **The volume of the pyramid shown below is 480 in^3. What do you think the formula to find the volume of a pyramid is? Explain your reasoning.**



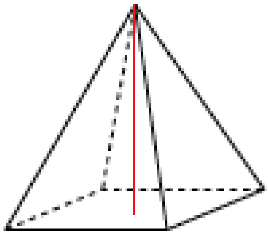
Since $480 = \frac{1440}{3}$, *the formula to find the volume of a pyramid is likely* $\frac{1}{3}Bh$, *where* B *is the area of the base. This is similar to the volume of a cone compared to the volume of a cylinder with the same base and height. The volume of a pyramid is* $\frac{1}{3}$ *of the volume of the rectangular prism with the same base and height.*

Discussion (5 minutes)

- What do you think the formula to find the volume of a pyramid is? Explain.

Ask students to share their response to part (b) of Exercise 2. If students do not see the connection between cones and cylinders to pyramids and prisms, then use the discussion points below.

- A pyramid is similar to a cone, but a pyramid has a rectangular base and faces that are shaped like triangles. For now we will focus on square pyramids only, that is, pyramids that have a base that is a square.

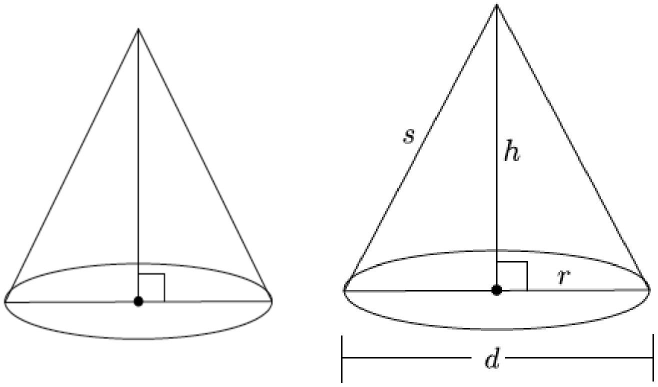


- The relationship between a cone and cylinder is similar for pyramids and prisms. How are the volumes of cones and cylinders related?
  - *A cone is one-third the volume of a cylinder with the same base and height.*
- In general, we say that the volume of a cylinder is $V = Bh$, where B is the area of the base. Then the volume of a cone is $V = \frac{1}{3}Bh$, again where B is the area of the base.
- How do you think the volumes of pyramids and rectangular prisms are related?
  - *The volume of a pyramid is one-third the volume of a rectangular prism with the same base and height.*
- In general, the volume of a rectangular prism is $V = Bh$, where B is the area of the base. Then the volume of a pyramid is $V = \frac{1}{3}Bh$, again where B is the area of the base.

Example 1

Example 1

State as many facts as you can about a cone.



$$A = \pi(r^2)$$
$$C = 2\pi(r) = \pi(d)$$
$$V = \frac{1}{3}\pi(r^2)h$$

Provide students with a minute or two to discuss as many facts as they can about a circular cone, and then have them share their facts with the class. As they identify parts of the cone and facts about the cone, label the drawing above. Students should be able to state or identify the following: radius, diameter, height, base, area of a circle is $A = \pi r^2$, circumference of a circle is $C = 2\pi r = 2d$, and the volume of a cone is $V = \frac{1}{3}Bh$, where B is the area of the base.

- What part of the cone have we not identified?
  - *The slanted part of the cone.*
- The slanted part of the cone is known as the lateral length, which is also referred to as the slant height. We denote the lateral length of a cone by s.

Label the lateral length of the cone with s on the drawing on the previous page.

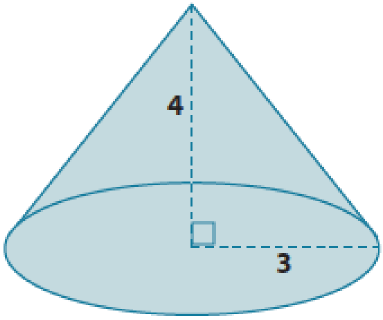
- Now that we know about the lateral length of a cone, we can begin using it in our work.

Exercises 3–6 (9 minutes)

Students work in pairs to complete Exercises 3–6. Students may need assistance determining the dimensions of the base of a pyramid. Let students reason through it first, offering guidance if necessary. Consider allowing students to use a calculator or to leave their answers as square roots, but not approximated unless asked, so as not to distract from the goal of the lesson. As needed, continue to remind students that we need only consider the positive square root of a number when our context involves length.

Exercises 3–10

3. What is the lateral length of the cone shown below?

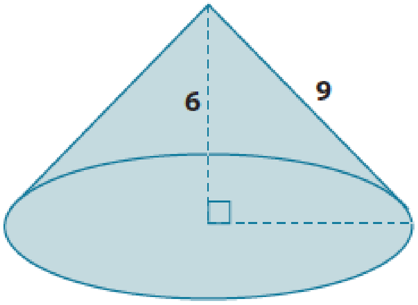


Let c be the lateral length.

$$3^2 + 4^2 = c^2$$
$$9 + 16 = c^2$$
$$25 = c^2$$
$$\sqrt{25} = \sqrt{c^2}$$
$$5 = c$$

The lateral length of the cone is 5 units.

4. Determine the exact volume of the cone shown below.



Let r be the radius of the base.

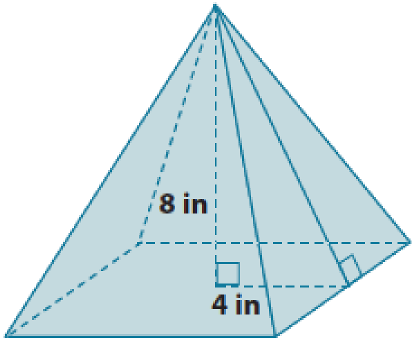
$$6^2 + r^2 = 9^2$$
$$36 + r^2 = 81$$
$$r^2 = 45$$

The area of the base is 45π units².

$$V = \frac{1}{3}Bh$$
$$V = \frac{1}{3}45\pi(6)$$
$$V = 90\pi$$

The volume of the cone is 90π units³.

5. **What is the lateral length (slant height) of the pyramid shown below? Give an exact square root answer and an approximate answer rounded to the tenths place.**

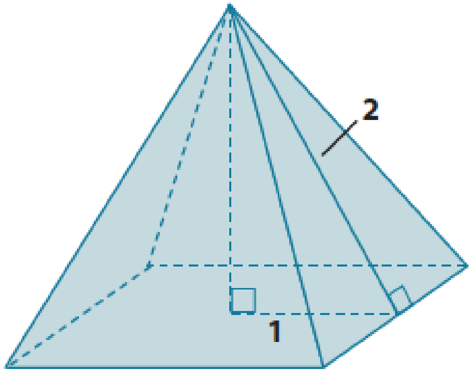


Let c represent the lateral length of the pyramid in inches.

$$\begin{aligned} 4^2 + 8^2 &= c^2 \\ 16 + 64 &= c^2 \\ 80 &= c^2 \\ \sqrt{80} &= \sqrt{c^2} \\ \sqrt{80} &= c \end{aligned}$$

The number $\sqrt{80}$ is between 8 and 9. In the sequence of tenths, it is between 8.9 and 9.0. Since 80 is closer to 8.9^2 than 9^2, the approximate lateral length is 8.9 inches.

6. **Determine the volume of the pyramid shown below. Give an exact answer using a square root.**



Let h be the height of the pyramid.

$$\begin{aligned} 1^2 + h^2 &= 2^2 \\ 1 + h^2 &= 4 \\ h^2 &= 3 \\ \sqrt{h^2} &= \sqrt{3} \\ h &= \sqrt{3} \end{aligned}$$

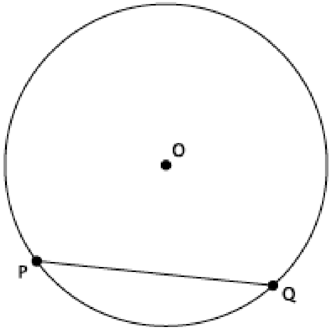
The area of the base is 4 units^2.

$$\begin{aligned} V &= \frac{1}{3}(4)(\sqrt{3}) \\ &= \frac{4\sqrt{3}}{3} \end{aligned}$$

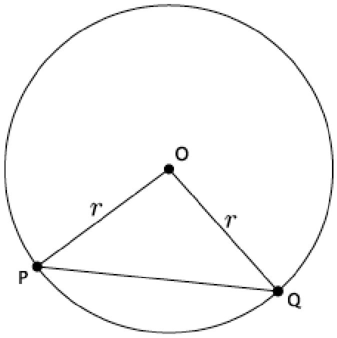
The volume of the pyramid is $\frac{4\sqrt{3}}{3}$ units^3.

Discussion (7 minutes)

- Let O be the center of a circle, and let P and Q be two points on the circle as shown. Then PQ is called a chord of the circle.



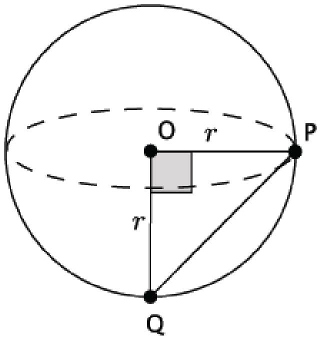
- What do you notice about the lengths $|OP|$ and $|OQ|$?
  - *Both lengths are equal to the radius, r, of the circle, which means they are equal in length to each other.*



- Will lengths $|OP|$ and $|OQ|$ always be equal to r, no matter where the chord is drawn?

Provide students time to place points P and Q around the circle to get an idea that no matter where the endpoints of the chord are placed, the length from the center of the circle to each of those points is always equal to r. The reason is based on the definition of a chord. Points P and Q must lie on the circle in order for PQ to be identified as a chord.

- When the angle $\angle POQ$ is a right angle, we can use the Pythagorean theorem to determine the length of the chord given the length of the radius; or, if we know the length of the chord, we can determine the length of the radius.
- Similarly, when points P and Q are on the surface of a sphere the segment that connects them is called a chord.

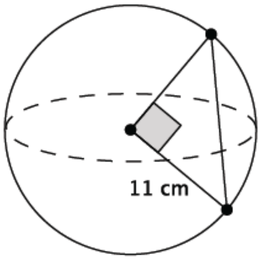


- Just like with circles, if the angle formed by POQ is a right angle, then we can use the Pythagorean theorem to find the length of the chord if we are given the length of the radius; or, given the length of the chord, we can determine the radius of the sphere.

Exercises 7–10 (9 minutes)

Students work in pairs to complete Exercises 7–10. Consider allowing students to use their calculators or to leave their answers as square roots (simplified square roots if that lesson was used with students), but not approximated, so as not to distract from the goal of the lesson.

7. **What is the length of the chord of the sphere shown below? Give an exact answer using a square root.**

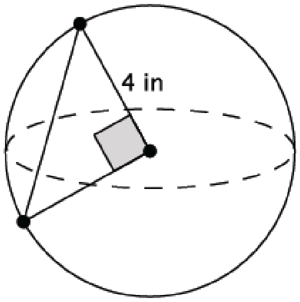


Let c represent the length of the chord in centimeters.

$$11^2 + 11^2 = c^2$$
$$121 + 121 = c^2$$
$$242 = c^2$$
$$\sqrt{242} = \sqrt{c^2}$$
$$\sqrt{11^2 \times 2} = c$$
$$11\sqrt{2} = c$$

The length of the chord is $\sqrt{242} = 11\sqrt{2}$ cm.

8. **What is the length of the chord of the sphere shown below? Give an exact answer using a square root.**

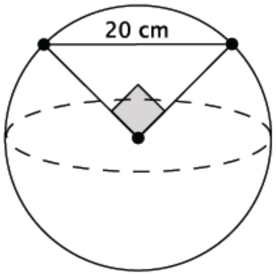


Let c represent the length of the chord in inches.

$$4^2 + 4^2 = c^2$$
$$16 + 16 = c^2$$
$$32 = c^2$$
$$\sqrt{32} = \sqrt{c^2}$$
$$\sqrt{4^2 \times 2} = c$$
$$4\sqrt{2} = c$$

The length of the chord is $\sqrt{32} = 4\sqrt{2}$ in.

9. **What is the volume of the sphere shown below? Give an exact answer using a square root.**



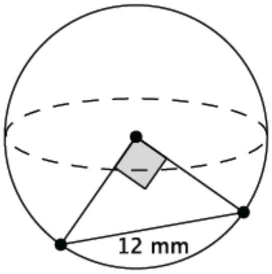
Let r represent the radius of the sphere in centimeters.

$$r^2 + r^2 = 20^2$$
$$2r^2 = 400$$
$$r^2 = 200$$
$$\sqrt{r^2} = \sqrt{200}$$
$$r = \sqrt{200}$$
$$r = \sqrt{10^2 \times 2}$$
$$r = 10\sqrt{2}$$

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi\left(10\sqrt{2}\right)^3$$
$$= \frac{4}{3}\pi 10^3\left(\sqrt{2}\right)^3$$
$$= \frac{4}{3}\pi 1000\left(\sqrt{8}\right)$$
$$= \frac{4}{3}\pi 1000\left(\sqrt{2^2 \times 2}\right)$$
$$= \frac{4}{3}\pi 1000(2)\left(\sqrt{2}\right)$$
$$= \frac{8000\sqrt{2}}{3}\pi$$

The volume of the sphere is $\frac{4}{3}\pi(\sqrt{200}\text{ cm})^3 = \frac{8000\sqrt{2}}{3}\pi\text{ cm}^3$.

10. **What is the volume of the sphere shown below? Give an exact answer using a square root.**



Let r represent the radius of the sphere in millimeters.

$$\begin{aligned} r^2 + r^2 &= 12^2 \\ 2r^2 &= 144 \\ r^2 &= 72 \\ \sqrt{r^2} &= \sqrt{72} \\ r &= \sqrt{6^2 \times 2} \\ r &= 6\sqrt{2} \end{aligned}$$

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi\left(6\sqrt{2}\right)^3 \\ &= \frac{4}{3}\pi 6^3\left(\sqrt{2}\right)^3 \\ &= \frac{4}{3}\pi 216\left(\sqrt{8}\right) \\ &= \frac{4}{3}\pi 216\left(\sqrt{2^2 \times 2}\right) \\ &= \frac{4}{3}\pi 216(2)\left(\sqrt{2}\right) \\ &= \frac{1728\sqrt{2}}{3}\pi \\ &= 576\,\pi\sqrt{2} \end{aligned}$$

The volume of the sphere is $576\,\pi\sqrt{2}\ \mathrm{mm}^3$.

Closing (5 minutes)

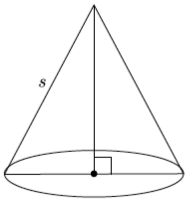
Summarize, or ask students to summarize, the main points from the lesson:

- The volume formulas for cones and cylinders are similar to those of pyramids and rectangular prisms.
- The formula to determine the volume of a pyramid is $\frac{1}{3}Bh$, where B is the area of the base. This is similar to the formula to determine the volume of a cone.
- The segment formed by two points on a circle is called a chord.
- We know how to apply the Pythagorean theorem to cones and spheres to determine volume.

Lesson Summary

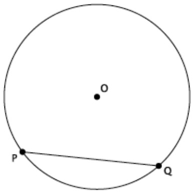
The volume formula for a right square pyramid is $V = \frac{1}{3}Bh$, where B is the area of the square base.

The lateral length of a cone, sometimes referred to as the slant height, is the side, s, shown in the diagram below.



Given the lateral length and the length of the radius, the Pythagorean theorem can be used to determine the height of the cone.

Let O be the center of a circle, and let P and Q be two points on the circle. Then PQ is called a chord of the circle.



The segments OP and OQ are equal in length because both represent the radius of the circle. If the angle formed by POQ is a right angle, then the Pythagorean theorem can be used to determine the length of the radius when given the length of the chord, or the length of the chord can be determined if given the length of the radius.

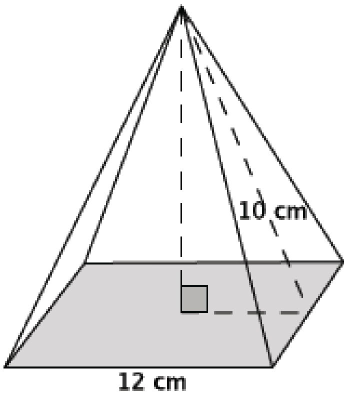
Exit Ticket (5 minutes)

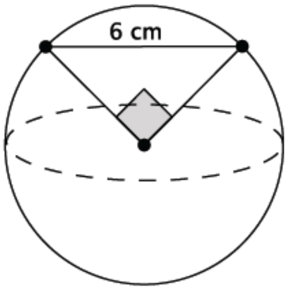
Name ______________________________________ Date ______________

Lesson 19: Cones and Spheres

Exit Ticket

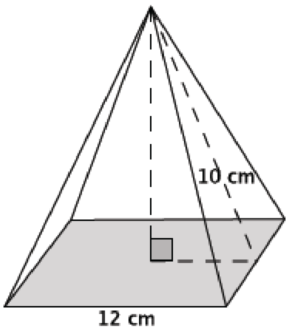
Which has the larger volume? Give an approximate answer rounded to the tenths place.

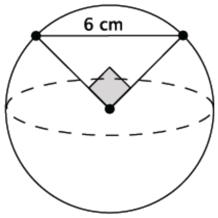




Exit Ticket Sample Solutions

Which has the larger volume? Give an approximate answer rounded to the tenths place.





Let h represent the height of the square pyramid, in centimeters.

$$h^2 + 6^2 = 10^2$$
$$h^2 + 36 = 100$$
$$h^2 = 64$$
$$h = 8$$

$$V = \frac{1}{3}(12^2)8$$
$$V = \frac{1}{3}(144)8$$
$$V = 384$$

The volume of the square pyramid is 384 cm^3.

Let r represent the radius of the sphere in centimeters.

$$r^2 + r^2 = 6^2$$
$$2r^2 = 36$$
$$r^2 = 18$$
$$\sqrt{r^2} = \sqrt{18}$$
$$r = \sqrt{3^2 \times 2}$$
$$r = 3\sqrt{2}$$

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi\left(3\sqrt{2}\right)^3 \\ &= \frac{4}{3}\pi 3^3\left(\sqrt{2}\right)^3 \\ &= \frac{4}{3}\pi 27\left(\sqrt{8}\right) \\ &= \frac{4}{3}\pi 27\left(\sqrt{2^2 \times 2}\right) \\ &= \frac{4}{3}\pi 27(2)\left(\sqrt{2}\right) \\ &= 72\pi\sqrt{2} \end{aligned}$$

The volume of the sphere is $72\pi\sqrt{2}\text{ cm}^3$.

The number $\sqrt{2}$ is between 1 and 2. In the sequence of tenths, it is between 1.4 and 1.5. Since 2 is closer to 1.4^2 than 1.5^2, then the number is approximately 1.4.

We know from previous lessons we can estimate $\pi = 3.1$.

Then, the approximate volume of the sphere is

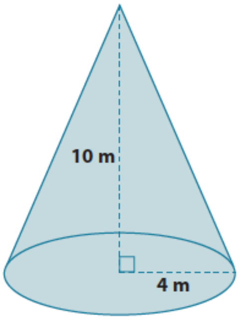
$$V = 72(1.4)3.1$$
$$V = 312.48$$

Therefore, the volume of the square pyramid is greater.

Problem Set Sample Solutions

Students use the Pythagorean theorem to solve mathematical problems in three dimensions.

1. **What is the lateral length of the cone shown below? Give an approximate answer rounded to the tenths place.**



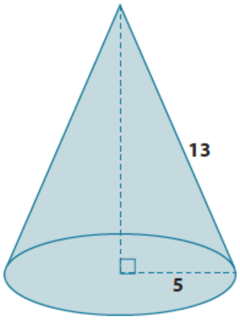
Let c be the lateral length in meters.

$$\begin{aligned} 10^2 + 4^2 &= c^2 \\ 100 + 16 &= c^2 \\ 116 &= c^2 \\ \sqrt{116} &= \sqrt{c^2} \\ \sqrt{116} &= c \end{aligned}$$

The number $\sqrt{116}$ is between 10 and 11. In the sequence of tenths, it is between 10.7 and 10.8. Since 116 is closer to 10.8^2 than 10.7^2, then the approximate value of the number is 10.8.

The lateral length of the cone is approximately 10.8 m.

2. **What is the volume of the cone shown below? Give an exact answer.**



Let h represent the height of a cone.

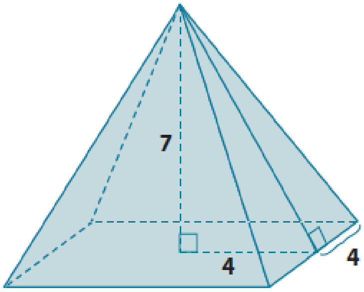
$$\begin{aligned} 5^2 + h^2 &= 13^2 \\ 25 + h^2 &= 169 \\ h^2 &= 144 \\ \sqrt{h^2} &= \sqrt{144} \\ h &= 12 \end{aligned}$$

The height of the cone is 12 units.

$$\begin{aligned} V &= \frac{1}{3}\pi\, 25(12) \\ &= 100\pi \end{aligned}$$

The volume of the cone is 100π units^3.

3. **Determine the volume and surface area of the pyramid shown below. Give exact answers.**



$$\begin{aligned} V &= \frac{1}{3}(64)(7) \\ &= \frac{448}{3} \end{aligned}$$

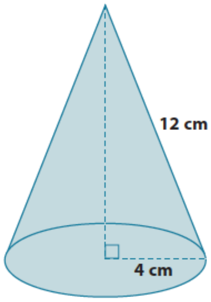
The volume of the pyramid is $\frac{448}{3}$ units^3.

Let c represent the lateral length.

$$\begin{aligned} 7^2 + 4^2 &= c^2 \\ 49 + 16 &= c^2 \\ 65 &= c^2 \\ \sqrt{65} &= \sqrt{c^2} \\ \sqrt{65} &= c \end{aligned}$$

The base area is 64 units^2, and the four faces are each $\frac{8\sqrt{65}}{2} = 4\sqrt{65}$, so the total area of the faces is $4 \times 4\sqrt{65} = 16\sqrt{65}$. The surface area is $64 + 16\sqrt{65}$ units^2.

4. **Alejandra computed the volume of the cone shown below as $64\pi\ \text{cm}^2$. Her work is shown below. Is she correct? If not, explain what she did wrong, and calculate the correct volume of the cone. Give an exact answer.**



$$V = \frac{1}{3}\pi(4^2)(12)$$
$$= \frac{16(12)\pi}{3}$$
$$= 64\pi$$
$$= 64\pi\ \text{cm}^3$$

Alejandra's work is incorrect. She used the lateral length instead of the height of the cone to compute volume.

Let h represent the height in centimeters.

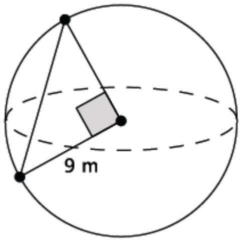
$$4^2 + h^2 = 12^2$$
$$16 + h^2 = 144$$
$$h^2 = 128$$
$$\sqrt{h^2} = \sqrt{128}$$
$$h = \sqrt{128}$$
$$h = \sqrt{8^2 \times 2}$$
$$h = 8\sqrt{2}$$

$$V = \frac{1}{3}\pi 4^2(8\sqrt{2})$$
$$V = \frac{1}{3}\pi 128\sqrt{2}$$
$$V = \frac{128\sqrt{2}}{3}\pi$$

The volume of the cone is $\frac{128\sqrt{2}}{3}\pi\ \text{cm}^3$.

5. **What is the length of the chord of the sphere shown below? Give an exact answer using a square root.**

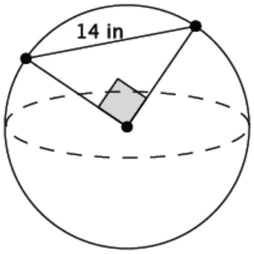
Let c represent the length of the chord in meters.



$$9^2 + 9^2 = c^2$$
$$81 + 81 = c^2$$
$$162 = c^2$$
$$\sqrt{162} = \sqrt{c^2}$$
$$\sqrt{162} = c$$
$$\sqrt{9^2 \times 2} = c$$
$$9\sqrt{2} = c$$

The length of the chord is $\sqrt{162} = 9\sqrt{2}$ m.

6. **What is the volume of the sphere shown below? Give an exact answer using a square root.**

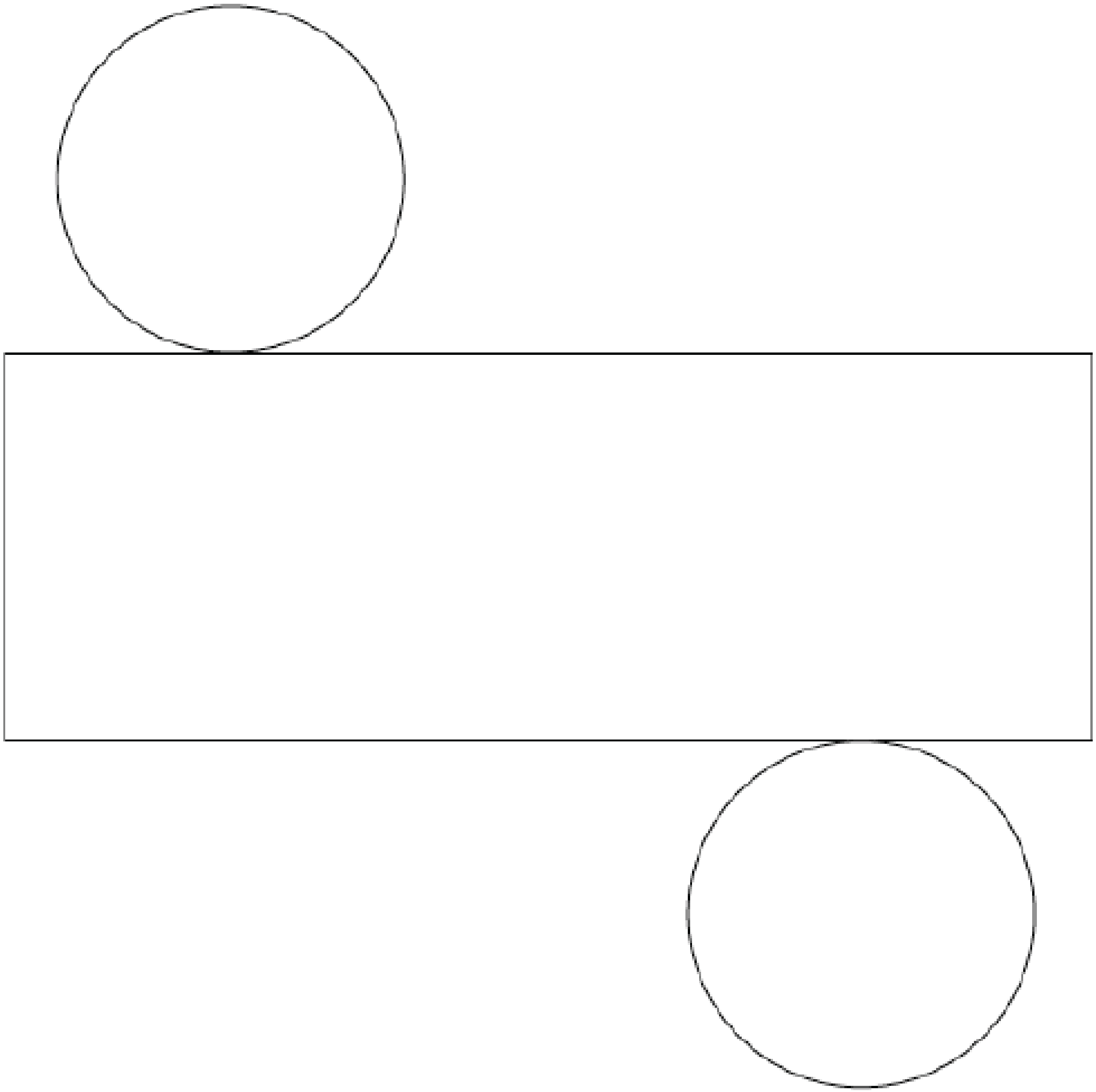


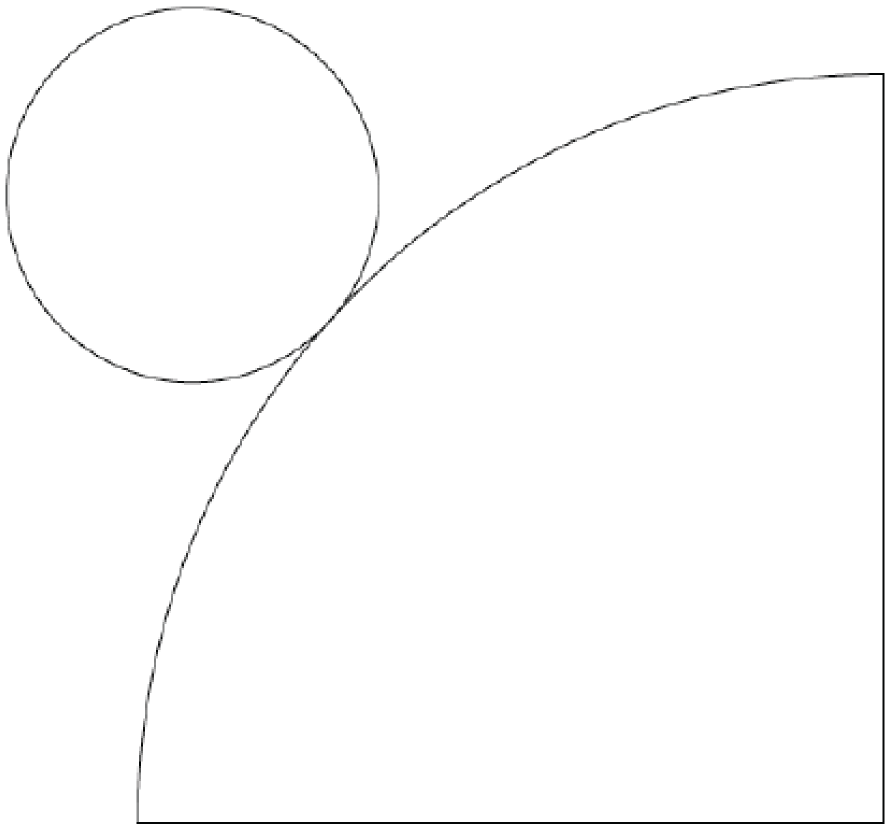
Let r represent the radius in inches.

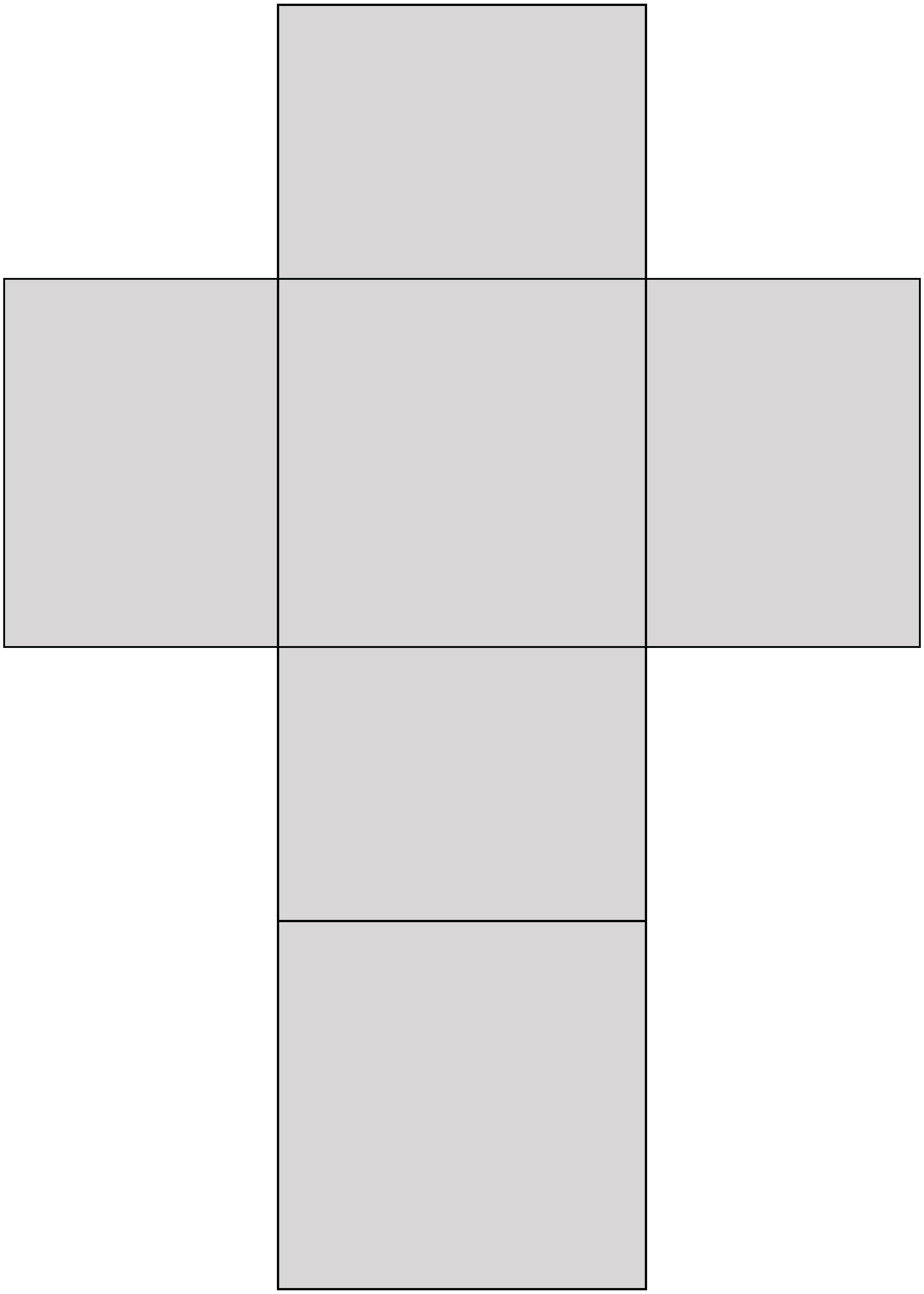
$$r^2 + r^2 = 14^2$$
$$2r^2 = 196$$
$$r^2 = 98$$
$$\sqrt{r^2} = \sqrt{98}$$
$$r = \sqrt{7^2 \times 2}$$
$$r = 7\sqrt{2}$$

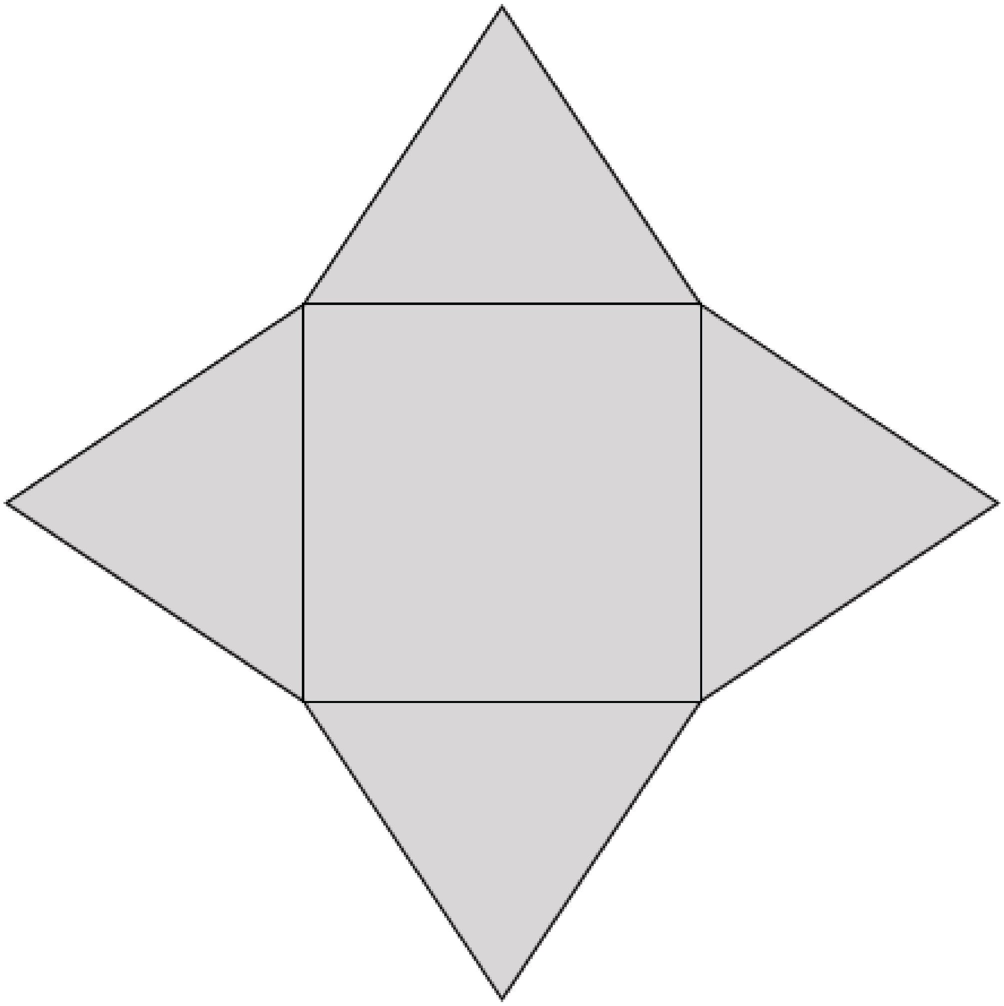
$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi\left(7\sqrt{2}\right)^3 \\ &= \frac{4}{3}\pi 343\left(\sqrt{8}\right) \\ &= \frac{4}{3}\pi 343\left(2\sqrt{2}\right) \\ &= \frac{2744\sqrt{2}}{3}\pi \end{aligned}$$

The volume of the sphere is $\frac{4}{3}\pi\left(\sqrt{98}\right)^3 = \frac{2744\sqrt{2}}{3}\pi\ \text{in}^3$.









Lesson 20: Truncated Cones

Student Outcomes

- Students know that truncated cones and pyramids are solids obtained by removing the top portion above a plane parallel to the base.
- Students find the volume of truncated cones.

Lesson Notes

Finding the volume of a truncated cone is not explicitly stated as part of the eighth-grade standards; however, finding the volume of a truncated cone combines two major skills learned in this grade, specifically, understanding similar triangles and their properties and calculating the volume of a cone. This topic is included because it provides an application of seemingly unrelated concepts. Furthermore, it allows students to see how learning one concept, similar triangles and their properties, can be applied to three-dimensional figures. Teaching this concept also reinforces students' understanding of similar triangles and how to determine unknown lengths of similar triangles.

Classwork

Opening Exercise (5 minutes)

Opening Exercise

1. **Examine the bucket below. It has a height of 9 inches and a radius at the top of the bucket of 4 inches.**

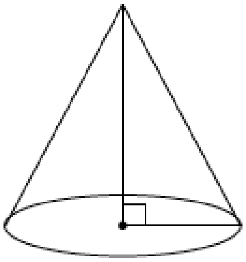
   a. **Describe the shape of the bucket. What is it similar to?**
   b. **Estimate the volume of the bucket.**

Discussion (10 minutes)

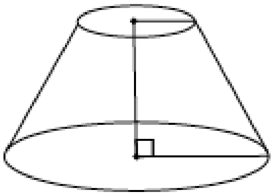
Before beginning the discussion, have students share their thoughts about the Opening Exercise. Students will likely say that the bucket is cone-shaped but not a cone or that it is cylinder-shaped but tapered. Any estimate between $48\pi \text{ in}^3$ (the volume of a cone with the given dimensions) and $144\pi \text{ in}^3$ (the volume of a cylinder with the given dimensions) is reasonable. Then, continue with the discussion below.

- When the top, narrower portion of a cone is removed such that the base of the removed portion is *parallel* to the existing base, the resulting shape is what we call a truncated cone.

  Here we have a cone:



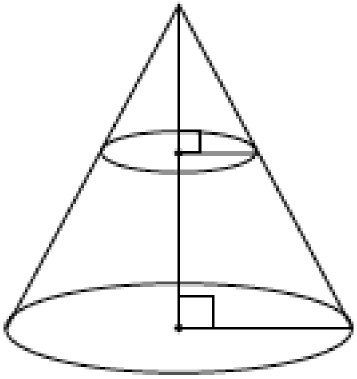
  Here we have a truncated cone:



  What is the shape of the removed portion?

  - *The removed portion of the figure will look like a cone. It will be a cone that is smaller than the original.*

- Here is the cone and the part that has been removed together in one drawing:



  Do you think the right triangles shown in the diagram are similar? Explain how you know.

Give students time to discuss the answer in groups, and then have them share their reasoning as to why the triangles are similar.

  - *Yes, the triangles are similar. Mark the top of the cone point O. Then, a dilation from O by scale factor r would map one triangle onto another. We also know that the triangles are similar because of the AA criterion. Each triangle has a right angle, and they have a common angle at the top of the cone (from the center of dilation).*

- What does that mean about the lengths of the legs and the hypotenuse of each right triangle?
  - *It means that the corresponding side lengths will be equal in ratio.*
- We will use all of these facts to help us determine the volume of a truncated cone.

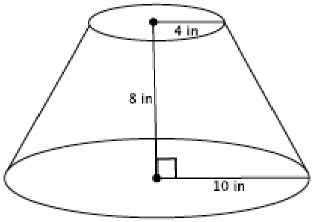
Example 1 (10 minutes)

MP.1

- Our goal is to determine the volume of the truncated cone shown below. Discuss in your groups how we might be able to do that.

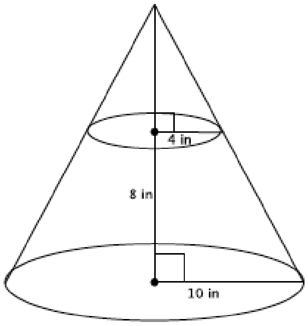
Example 1

Determine the volume of the truncated cone shown below.



Provide students time to discuss in groups a strategy for finding the volume of the truncated cone. Use the discussion questions below to guide their thinking as needed.

- Since we know that the original cone and the portion that has been removed to make this truncated cone are similar, let's begin by drawing in the missing portion.



- We know the formula to find the volume of a cone. Is there enough information in the new diagram for us to find the volume? Explain.
  - *No, there's not enough information. We would have to know the height of the cone, and at this point we only know the height of the truncated cone,* 8 *inches.*

MP.2

- Recall our conversation about the similar right triangles. We can use what we know about similarity to determine the height of the cone with the following proportion. What does each part of the proportion represent in the diagram?

$$\frac{4}{10} = \frac{x}{x+8}$$

  - *The* 4 *is the radius of the small cone. The* 10 *is the radius of the large cone. The* x *represents the height of the small cone in inches. The expression* $x + 8$ *represents the height of the large cone.*
- Work in your groups to determine the height of the small cone.
  - *Since the triangles are similar, we will let* x *represent the height of the cone that has been removed. Then,*

$$\begin{aligned} 4(x+8) &= 10x \\ 4x + 32 &= 10x \\ 32 &= 6x \\ \frac{32}{6} &= x \\ 5.\overline{3} &= x. \end{aligned}$$

- Now that we know the height of the cone that has been removed, we also know the total height of the cone. How might we use these pieces of information to determine the volume of the truncated cone?
  - *We can find the volume of the large cone, find the volume of the small cone that was removed, and then subtract the volumes. What will be left is the volume of the truncated cone.*
- Write an expression that represents the volume of the truncated cone. Use approximations for the heights since both are infinite decimals. Be prepared to explain what each part of the expression represents in the situation.
  - *The volume of the truncated cone is given by the expression*

$$\frac{1}{3}\pi 10^2(13.3) - \frac{1}{3}\pi 4^2(5.3)$$

where $\frac{1}{3}\pi 10^2(13.3)$ *is the volume of the large cone, and* $\frac{1}{3}\pi 4^2(5.3)$ *is the volume of the smaller cone. The difference in the volumes will be the volume of the truncated cone.*

- Determine the volume of the truncated cone. Use the approximate value of the number $5.\overline{3}$ when you compute the volumes.
  - *The volume of the small cone is*

$$\begin{aligned} V &= \frac{1}{3}\pi 4^2(5.3) \\ &= \frac{1}{3}\pi 84.8 \\ &= \frac{84.8}{3}\pi. \end{aligned}$$

The volume of the large cone is

$$\begin{aligned} V &= \frac{1}{3}\pi 10^2(13.3) \\ &= \frac{1330}{3}\pi. \end{aligned}$$

The volume of the truncated cone is

$$\frac{1330}{3}\pi - \frac{84.8}{3}\pi = \left(\frac{1330}{3} - \frac{84.8}{3}\right)\pi$$
$$= \frac{1245.2}{3}\pi.$$

The volume of the truncated cone is $\frac{1245.2}{3}\pi \text{ in}^3$.

MP.2

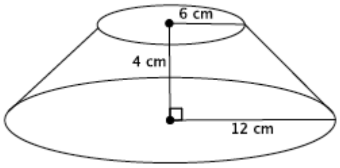
- Write an equivalent expression for the volume of a truncated cone that shows the volume is $\frac{1}{3}$ of the difference between two cylinders. Explain how your expression shows this.
  - *The expression* $\frac{1}{3}\pi 10^2(13.3) - \frac{1}{3}\pi 4^2(5.3)$ *can be written as* $\frac{1}{3}\left(\pi 10^2(13.3) - \pi 4^2(5.3)\right)$*, where* $\pi 10^2(13.3)$ *is the volume of the larger cylinder, and* $\pi 4^2(5.3)$ *is the volume of the smaller cylinder. One-third of the difference is the volume of a truncated cone with the same base and height measurements as the cylinders.*

Exercises 2–6 (10 minutes)

Students work in pairs or small groups to complete Exercises 2–6.

Exercises 2–6

2. Find the volume of the truncated cone.



a. Write a proportion that will allow you to determine the height of the cone that has been removed. Explain what all parts of the proportion represent.

$$\frac{6}{12} = \frac{x}{x+4}$$

Let x represent the height of the small cone in centimeters. Then, $x + 4$ is the height of the large cone (with the removed part included). The 6 represents the base radius of the removed cone, and the 12 is the base radius of the large cone.

b. Solve your proportion to determine the height of the cone that has been removed.

$$6(x+4) = 12x$$
$$6x + 24 = 12x$$
$$24 = 6x$$
$$4 = x$$

c. Write an expression that can be used to determine the volume of the truncated cone. Explain what each part of the expression represents.

$$\frac{1}{3}\pi 12^2(8) - \frac{1}{3}\pi 6^2(4)$$

The expression $\frac{1}{3}\pi 12^2(8)$ is the volume of the large cone, and $\frac{1}{3}\pi 6^2(4)$ is the volume of the small cone. The difference of the volumes gives the volume of the truncated cone.

d. **Calculate the volume of the truncated cone.**

The volume of the small cone is

$$V = \frac{1}{3}\pi 6^2(4) = \frac{144}{3}\pi.$$

The volume of the large cone is

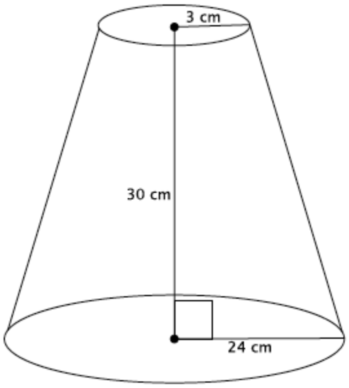
$$V = \frac{1}{3}\pi 12^2(8) = \frac{1152}{3}\pi.$$

The volume of the truncated cone is

$$\frac{1152}{3}\pi - \frac{144}{3}\pi = \left(\frac{1152}{3} - \frac{144}{3}\right)\pi = \frac{1008}{3}\pi = 336\pi.$$

The volume of the truncated cone is $336\pi\ \text{cm}^3$.

3. **Find the volume of the truncated cone.**



Let x represent the height of the small cone in centimeters.

$$\begin{aligned} \frac{3}{24} &= \frac{x}{x+30} \\ 3x + 90 &= 24x \\ 90 &= 21x \\ \frac{30}{7} &= x \\ 4.3 &\approx x \end{aligned}$$

The volume of the small cone is

$$V = \frac{1}{3}\pi 3^2(4.3) = \frac{38.7}{3}\pi = 12.9\pi.$$

The volume of the large cone is

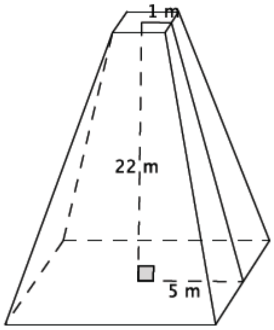
$$V = \frac{1}{3}\pi 24^2(34.3) = \frac{19756.8}{3}\pi = 6585.6\pi.$$

The volume of the truncated cone is

$$6585.6\pi - 12.9\pi = (6585.6 - 12.9)\pi = 6572.7\pi.$$

The volume of the truncated cone is $6572.7\pi\ \text{cm}^3$.

4. **Find the volume of the truncated pyramid with a square base.**



a. **Write a proportion that will allow you to determine the height of the cone that has been removed. Explain what all parts of the proportion represent.**

$$\frac{1}{5} = \frac{x}{x+22}$$

Let x represent the height of the small pyramid in meters. Then $x+22$ is the height of the large pyramid. The 1 represents half of the length of the base of the small pyramid, and the 5 represents half of the length of the base of the large pyramid.

b. **Solve your proportion to determine the height of the pyramid that has been removed.**

$$\begin{aligned} x+22 &= 5x \\ 22 &= 4x \\ 5.5 &= x \end{aligned}$$

c. **Write an expression that can be used to determine the volume of the truncated pyramid. Explain what each part of the expression represents.**

$$\frac{1}{3}(100)27.5 - \frac{1}{3}(4)5.5$$

The expression $\frac{1}{3}(100)27.5$ is the volume of the large pyramid, and $\frac{1}{3}(4)5.5$ is the volume of the small pyramid. The difference of the volumes gives the volume of the truncated pyramid.

d. **Calculate the volume of the truncated pyramid.**

The volume of the small pyramid is

$$\begin{aligned} V &= \frac{1}{3}(4)5.5 \\ &= \frac{22}{3}. \end{aligned}$$

The volume of the large pyramid is

$$\begin{aligned} V &= \frac{1}{3}(100)27.5 \\ &= \frac{2750}{3}. \end{aligned}$$

The volume of the truncated pyramid is

$$\frac{2750}{3} - \frac{22}{3} = \frac{2728}{3}.$$

The volume of the truncated pyramid is $\frac{2728}{3}$ m^3.

5. **A pastry bag is a tool used to decorate cakes and cupcakes. Pastry bags take the form of a truncated cone when filled with icing. What is the volume of a pastry bag with a height of 6 inches, large radius of 2 inches, and small radius of 0.5 inches?**

   Let x represent the height of the small cone in inches.

$$\frac{x}{x+6}=\frac{0.5}{2}$$
$$2x = 0.5(x+6)$$
$$2x=\frac{1}{2}x+3$$
$$\frac{3}{2}x=3$$
$$x=2$$

   The volume of the small cone is

$$V=\frac{1}{3}\pi\left(\frac{1}{2}\right)^2(2)$$
$$=\frac{1}{6}\pi.$$

   The volume of the large cone is

$$V=\frac{1}{3}\pi 2^2(8)$$
$$=\frac{32}{3}\pi.$$

   The volume of the truncated cone is

$$\frac{32}{3}\pi-\frac{1}{6}\pi=\left(\frac{32}{3}-\frac{1}{6}\right)\pi$$
$$=\frac{63}{6}\pi=\frac{21}{2}\pi.$$

   The volume of the pastry bag is $\frac{21}{2}\pi\ \text{in}^3$ *when filled.*

6. **Explain in your own words what a truncated cone is and how to determine its volume.**

   A truncated cone is a cone with a portion of the top cut off. The base of the portion that is cut off needs to be parallel to the base of the original cone. Since the portion that is cut off is in the shape of a cone, then to find the volume of a truncated cone, you must find the volume of the cone (without any portion cut off), find the volume of the cone that is cut off, and then find the difference between the two volumes. That difference is the volume of the truncated cone.

Closing (5 minutes)

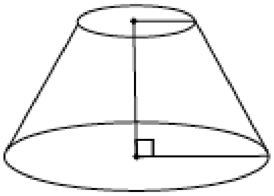
Summarize, or ask students to summarize, the main points from the lesson:

- A truncated cone or pyramid is a solid figure that is obtained by removing the top portion above a plane parallel to the base.
- Information about similar triangles can provide the information we need to determine the volume of a truncated figure.
- To find the volume of a truncated cone, first find the volume of the part of the cone that was removed, then the total volume of the cone. Finally, subtract the removed cone's volume from the total cone's volume. What is leftover is the volume of the truncated cone.

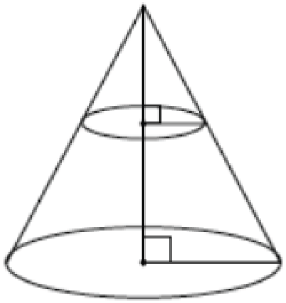
Lesson Summary

A truncated cone or pyramid is a solid figure that is obtained by removing the top portion above a plane parallel to the base. Shown below on the left is a truncated cone. A truncated cone with the top portion still attached is shown below on the right.

Truncated cone:



Truncated cone with top portion attached:



To determine the volume of a truncated cone, you must first determine the height of the portion of the cone that has been removed using ratios that represent the corresponding sides of the right triangles. Next, determine the volume of the portion of the cone that has been removed and the volume of the truncated cone with the top portion attached. Finally, subtract the volume of the cone that represents the portion that has been removed from the complete cone. The difference represents the volume of the truncated cone.

Pictorially,

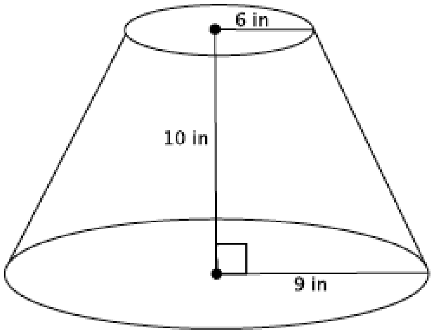
Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 20: Truncated Cones

Exit Ticket

Find the volume of the truncated cone.



a. Write a proportion that will allow you to determine the height of the cone that has been removed. Explain what all parts of the proportion represent.

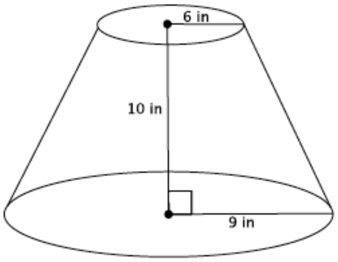
b. Solve your proportion to determine the height of the cone that has been removed.

c. Write an expression that can be used to determine the volume of the truncated cone. Explain what each part of the expression represents.

d. Calculate the volume of the truncated cone.

Exit Ticket Sample Solutions

Find the volume of the truncated cone.



a. **Write a proportion that will allow you to determine the height of the cone that has been removed. Explain what all parts of the proportion represent.**

$$\frac{6}{9}=\frac{x}{x+10}$$

Let x represent the height of the small cone in inches. Then $x+10$ is the height of the large cone. Then 6 is the base radius of the small cone, and the 9 is the base radius of the large cone.

b. **Solve your proportion to determine the height of the cone that has been removed.**

$$\begin{aligned}6(x+10)&=9x\\6x+60&=9x\\60&=3x\\20&=x\end{aligned}$$

c. **Write an expression that can be used to determine the volume of the truncated cone. Explain what each part of the expression represents.**

$$\frac{1}{3}\pi 9^2(30)-\frac{1}{3}\pi 6^2(20)$$

The expression $\frac{1}{3}\pi 9^2(30)$ represents the volume of the large cone, and $\frac{1}{3}\pi 6^2(20)$ is the volume of the small cone. The difference in volumes represents the volume of the truncated cone.

d. **Calculate the volume of the truncated cone.**

The volume of the small cone is

$$\begin{aligned}V&=\frac{1}{3}\pi 6^2(20)\\&=\frac{720}{3}\pi.\end{aligned}$$

The volume of the large cone is

$$\begin{aligned}V&=\frac{1}{3}\pi 9^2(30)\\&=\frac{2,430}{3}\pi.\end{aligned}$$

The volume of the truncated cone is

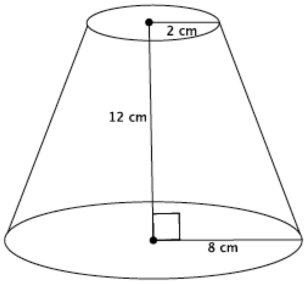
$$\begin{aligned}\frac{2430}{3}\pi-\frac{720}{3}\pi&=\left(\frac{2430}{3}-\frac{720}{3}\right)\pi\\&=\frac{1710}{3}\pi\\&=570\pi.\end{aligned}$$

The volume of the truncated cone is $570\pi\ \text{in}^3$.

Problem Set Sample Solutions

Students use what they know about similar triangles to determine the volume of truncated cones.

1. **Find the volume of the truncated cone.**



a. **Write a proportion that will allow you to determine the height of the cone that has been removed. Explain what each part of the proportion represents.**

$$\frac{2}{8}=\frac{x}{x+12}$$

Let x represent the height of the small cone in cm. Then $x+12$ is the height of the large cone. The 2 represents the base radius of the small cone, and the 8 represents the base radius of the large cone.

b. **Solve your proportion to determine the height of the cone that has been removed.**

$$\begin{aligned}2(x+12)&=8x\\2x+24&=8x\\24&=6x\\4&=x\end{aligned}$$

c. **Show a fact about the volume of the truncated cone using an expression. Explain what each part of the expression represents.**

$$\frac{1}{3}\pi 8^2(16)-\frac{1}{3}\pi 2^2(4)$$

The expression $\frac{1}{3}\pi 8^2(16)$ represents the volume of the large cone, and $\frac{1}{3}\pi 2^2(4)$ is the volume of the small cone. The difference in volumes gives the volume of the truncated cone.

d. **Calculate the volume of the truncated cone.**

The volume of the small cone is

$$\begin{aligned}V&=\frac{1}{3}\pi 2^2(4)\\&=\frac{16}{3}\pi.\end{aligned}$$

The volume of the large cone is

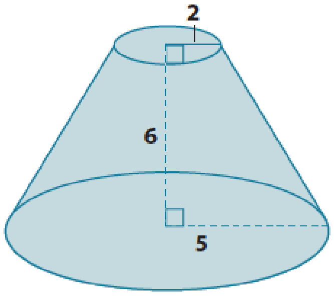
$$\begin{aligned}V&=\frac{1}{3}\pi 8^2(16)\\&=\frac{1024}{3}\pi.\end{aligned}$$

The volume of the truncated cone is

$$\begin{aligned}\frac{1024}{3}\pi-\frac{16}{3}\pi&=\left(\frac{1024}{3}-\frac{16}{3}\right)\pi\\&=\frac{1008}{3}\pi\\&=336\pi.\end{aligned}$$

The volume of the truncated cone is $336\pi\ \text{cm}^3$.

2. Find the volume of the truncated cone.



Let x represent the height of the small cone.

$$\frac{2}{5} = \frac{x}{x+6}$$
$$2(x+6) = 5x$$
$$2x + 12 = 5x$$
$$12 = 3x$$
$$4 = x$$

The volume of the small cone is

$$V = \frac{1}{3}\pi 2^2(4)$$
$$= \frac{16}{3}\pi.$$

The volume of the large cone is

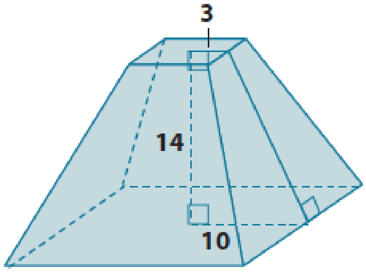
$$V = \frac{1}{3}\pi 5^2(10)$$
$$= \frac{250}{3}\pi.$$

The volume of the truncated cone is

$$\frac{250}{3}\pi - \frac{16}{3}\pi = \left(\frac{250}{3} - \frac{16}{3}\right)\pi$$
$$= \frac{234}{3}\pi$$
$$= 78\pi.$$

The volume of the truncated cone is 78π *units*^{3}*.*

3. Find the volume of the truncated pyramid with a square base.



Let x represent the height of the small pyramid.

$$\frac{3}{10} = \frac{x}{x+14}$$
$$3(x+14) = 10x$$
$$3x + 42 = 10x$$
$$42 = 7x$$
$$6 = x$$

The volume of the small pyramid is

$$V = \frac{1}{3}(36)(6)$$
$$= \frac{216}{3}.$$

The volume of the large pyramid is

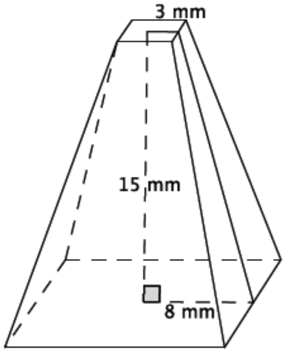
$$V = \frac{1}{3}(400)(20)$$
$$= \frac{8000}{3}.$$

The volume of the truncated pyramid is

$$\frac{8000}{3} - \frac{216}{3} = \frac{7784}{3}.$$

The volume of the truncated pyramid is $\frac{7784}{3}$ *units*^{3}*.*

4. **Find the volume of the truncated pyramid with a square base. Note: 3 mm is the distance from the center to the edge of the square at the top of the figure.**



Let x represent the height of the small pyramid in millimeters.

$$\frac{3}{8} = \frac{x}{x+15}$$
$$3(x+15) = 8x$$
$$3x + 45 = 8x$$
$$45 = 5x$$
$$9 = x$$

The volume of the small pyramid is

$$V = \frac{1}{3}(36)(9) = 108.$$

The volume of the large pyramid is

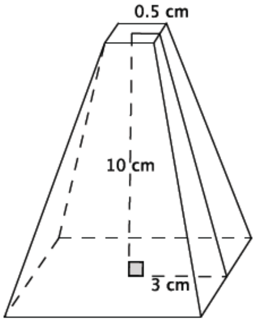
$$V = \frac{1}{3}(256)(24) = 2048.$$

The volume of the truncated pyramid is

$$2048 - 108 = 1940.$$

The volume of the truncated pyramid is 1940 mm^3.

5. **Find the volume of the truncated pyramid with a square base. Note: 0.5 cm is the distance from the center to the edge of the square at the top of the figure.**



Let x represent the height of the small pyramid in centimeters.

$$\frac{0.5}{3} = \frac{x}{x+10}$$
$$\frac{1}{2}(x+10) = 3x$$
$$\frac{1}{2}x + 5 = 3x$$
$$5 = \frac{5}{2}x$$
$$2 = x$$

The volume of the small pyramid is

$$V = \frac{1}{3}(1)(2) = \frac{2}{3}.$$

The volume of the large pyramid is

$$V = \frac{1}{3}(36)(12) = \frac{432}{3}.$$

The volume of the truncated pyramid is

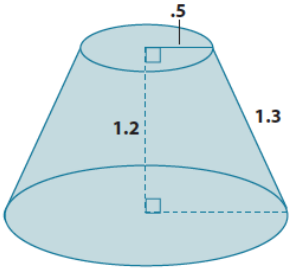
$$\frac{432}{3} - \frac{2}{3} = \frac{430}{3}.$$

The volume of the truncated pyramid is $\frac{430}{3}\text{ cm}^3$.

6. **Explain how to find the volume of a truncated cone.**

The first thing you have to do is use the ratios of corresponding sides of similar triangles to determine the height of the cone that was removed to make the truncated cone. Once you know the height of that cone, you can determine its volume. Then, you can find the height of the cone (the truncated cone and the portion that was removed). Once you know both volumes, you can subtract the smaller volume from the larger volume. The difference is the volume of the truncated cone.

7. **Challenge: Find the volume of the truncated cone.**



Since the height of the truncated cone is 1.2 units, we can drop a perpendicular line from the top of the cone to the bottom of the cone so that we have a right triangle with a leg length of 1.2 units and a hypotenuse of 1.3 units. Then, by the Pythagorean theorem, if b is the length of the leg of the right triangle, then

$$\begin{aligned} 1.2^2 + b^2 &= 1.3^2 \\ 1.44 + b^2 &= 1.69 \\ b^2 &= 0.25 \\ b &= 0.5. \end{aligned}$$

The part of the radius of the bottom base found by the Pythagorean theorem is 0.5. When we add the length of the upper radius (because if you translate along the height of the truncated cone, then it is equal to the remaining part of the lower base), then the radius of the lower base is 1.

Let x represent the height of the small cone.

$$\begin{aligned} \frac{0.5}{1} &= \frac{x}{x+1.2} \\ \frac{1}{2}(x+1.2) &= x \\ \frac{1}{2}x + 0.6 &= x \\ 0.6 &= \frac{1}{2}x \\ 1.2 &= x \end{aligned}$$

The volume of the small cone

$$\begin{aligned} V &= \frac{1}{3}\pi(0.5^2)(1.2) \\ &= \frac{0.3}{3}\pi \\ &= 0.1\pi. \end{aligned}$$

The volume of the large cone is

$$\begin{aligned} V &= \frac{1}{3}\pi(1^2)(2.4) \\ &= \frac{2.4}{3}\pi \\ &= 0.8\pi. \end{aligned}$$

The volume of the truncated cone is

$$\begin{aligned} \frac{2.4}{3}\pi - \frac{0.3}{3}\pi &= \left(\frac{2.4}{3} - \frac{0.3}{3}\right)\pi \\ &= \frac{2.1}{3}\pi \\ &= 0.7\pi. \end{aligned}$$

The volume of the truncated cone is 0.7π units^3.

Lesson 21: Volume of Composite Solids

Student Outcomes

- Students know how to determine the volume of a figure composed of combinations of cylinders, cones, and spheres.

Classwork

Exploratory Challenge/Exercises 1–4 (20 minutes)

Students should know that volumes can be added as long as the solids touch only on the boundaries of their figures. That is, there cannot be any overlapping sections. Students should understand this with the first exercise. Then, allow them to work independently or in pairs to determine the volumes of composite solids in Exercises 1–4. All of the exercises include MP.1, where students persevere with some challenging problems and compare solution methods, and MP.2, where students explain how the structure of their expressions relate to the diagrams from which they were created.

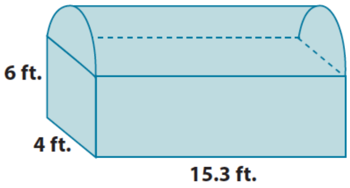
Exercises 1–4

1.

a. Write an expression that can be used to find the volume of the chest shown below. Explain what each part of your expression represents.

$$(4 \times 15.3 \times 6) + \frac{1}{2}\left(\pi 2^2(15.3)\right)$$

The expression $(4 \times 15.3 \times 6)$ *represents the volume of the prism, and* $\frac{1}{2}\left(\pi 2^2(15.3)\right)$ *is the volume of the half-cylinder on top of the chest. Adding the volumes together will give the total volume of the chest.*



b. What is the approximate volume of the chest shown above? Use 3.14 for an approximation of π. Round your final answer to the tenths place.

The rectangular prism at the bottom has the following volume:

$$\begin{aligned} V &= 4 \times 15.3 \times 6 \\ &= 367.2 \end{aligned}$$

The half-cylinder top has the following volume:

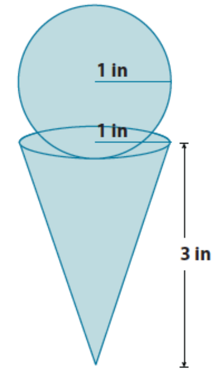
$$\begin{aligned} V &= \frac{1}{2}\left(\pi 2^2(15.3)\right) \\ &= \frac{1}{2}(61.2\pi) \\ &= 30.6\pi \\ &= 96.084 \end{aligned}$$

The total volume of the chest shown is approximately $367.2\text{ ft}^3 + 96.084\text{ ft}^3 = 463.284\text{ ft}^3 \approx 463.3\text{ ft}^3$.

Once students have finished the first exercise, ask them what they noticed about the total volume of the chest and what they noticed about the boundaries of each figure that comprised the shape of the chest. These questions illustrate the key understanding that volume is additive as long as the solids touch only at the boundaries and do not overlap.

2.

a. Write an expression for finding the volume of the figure shown below. Explain what each part of your expression represents.



$$\frac{4}{3}\pi 1^3 + \frac{1}{3}\pi 1^2(3)$$

The expression $\frac{4}{3}\pi 1^3$ *represents the volume of the sphere, and* $\frac{1}{3}\pi 1^2(3)$ *represents the volume of the cone. The sum of those two expressions gives the total volume of the figure.*

b. Assuming every part of the cone can be filled with ice cream, what is the exact and approximate volume of the cone and scoop? (Recall that exact answers are left in terms of π, and approximate answers use 3.14 for π). Round your approximate answer to the hundredths place.

The volume of the scoop is

$$\begin{aligned} V &= \frac{4}{3}\pi 1^3 \\ &= \frac{4}{3}\pi \\ &\approx 4.19. \end{aligned}$$

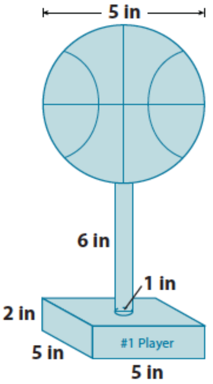
The volume of the cone is

$$\begin{aligned} V &= \frac{1}{3}\pi 1^2(3) \\ &= \pi \\ &\approx 3.14. \end{aligned}$$

The total volume of the cone and scoop is approximately $4.19 \text{ in}^3 + 3.14 \text{ in}^3 = 7.33 \text{ in}^3$. *The exact volume of the cone and scoop is* $\frac{4}{3}\pi^3 \text{ in}^3 + \pi^3 \text{ in}^3 = \frac{7}{3}\pi \text{ in}^3$.

3.

a. Write an expression for finding the volume of the figure shown below. Explain what each part of your expression represents.



$$(5 \times 5 \times 2) + \pi\left(\frac{1}{2}\right)^2(6) + \frac{4}{3}\pi(2.5)^3$$

The expression $(5 \times 5 \times 2)$ *represents the volume of the rectangular base,* $\pi\left(\frac{1}{2}\right)^2(6)$ *represents the volume of the cylinder, and* $\frac{4}{3}\pi(2.5)^3$ *is the volume of the sphere on top. The sum of the separate volumes gives the total volume of the figure.*

b. Every part of the trophy shown is made out of silver. How much silver is used to produce one trophy? Give an exact and approximate answer rounded to the hundredths place.

The volume of the rectangular base is

$$\begin{aligned} V &= 5 \times 5 \times 2 \\ &= 50. \end{aligned}$$

The volume of the cylinder holding up the basketball is

$$\begin{aligned} V &= \pi\left(\frac{1}{2}\right)^2(6) \\ &= \frac{1}{4}\pi(6) \\ &= \frac{3}{2}\pi \\ &\approx 4.71. \end{aligned}$$

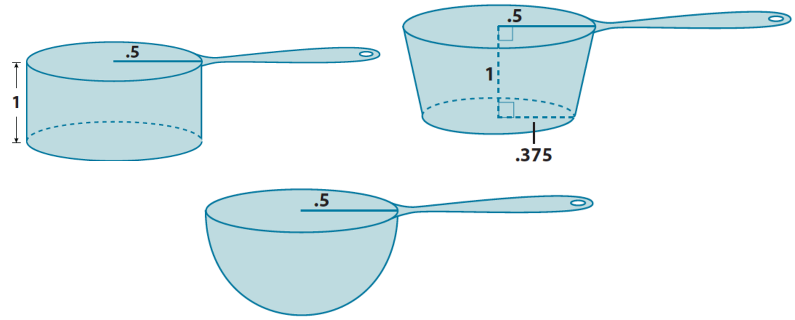
The volume of the basketball is

$$\begin{aligned} V &= \frac{4}{3}\pi(2.5)^3 \\ &= \frac{4}{3}\pi(15.625) \\ &= \frac{62.5}{3}\pi \\ &\approx 65.42. \end{aligned}$$

The approximate total volume of silver needed is $50 \text{ in}^3 + 4.71 \text{ in}^3 + 65.42 \text{ in}^3 = 120.13 \text{ in}^3$. *The exact volume of the trophy is* $50 \text{ in}^3 + \frac{3}{2}\pi \text{ in}^3 + \frac{62.5}{3}\pi \text{ in}^3 = 50 \text{ in}^3 + \left(\frac{3}{2} + \frac{62.5}{3}\right)\pi \text{ in}^3 = 50 \text{ in}^3 + \frac{134}{6}\pi \text{ in}^3 = 50 \text{ in}^3 + \frac{67}{3}\pi \text{ in}^3$.

4. **Use the diagram of scoops below to answer parts (a) and (b).**

   a. **Order the scoops from least to greatest in terms of their volumes. Each scoop is measured in inches.**



The volume of the cylindrical scoop is

$$V = \pi\left(\frac{1}{2}\right)^2(1)$$
$$= \frac{1}{4}\pi.$$

The volume of the spherical scoop is

$$V = \frac{1}{2}\left(\frac{4}{3}\pi\left(\frac{1}{2}\right)^3\right)$$
$$= \frac{1}{2}\left(\frac{4}{3}\pi\left(\frac{1}{8}\right)\right)$$
$$= \frac{4}{48}\pi$$
$$= \frac{1}{12}\pi.$$

The volume of the truncated cone scoop is as follows.

Let x represent the height of the portion of the cone that was removed.

$$\frac{0.5}{0.375} = \frac{x+1}{x}$$
$$0.5x = 0.375(x+1)$$
$$0.5x = 0.375x + 0.375$$
$$0.125x = 0.375$$
$$x = 3$$

The volume of the small cone is

$$V = \frac{1}{3}\pi(0.375)^2(3)$$
$$= \frac{9}{64}\pi.$$

The volume of the large cone is

$$V = \frac{1}{3}\pi(0.5)^2(4)$$
$$= \frac{1}{3}\pi.$$

The volume of the truncated cone is

$$\frac{1}{3}\pi - \frac{9}{64}\pi = \left(\frac{1}{3} - \frac{9}{64}\right)\pi$$
$$= \frac{64-27}{192}\pi$$
$$= \frac{37}{192}\pi.$$

The three scoops have volumes of $\frac{1}{4}\pi\text{ in}^3$, $\frac{1}{12}\pi\text{ in}^3$, *and* $\frac{37}{192}\pi\text{ in}^3$. *In order from least to greatest, they are* $\frac{1}{12}\pi\text{ in}^3$, $\frac{37}{192}\pi\text{ in}^3$, *and* $\frac{1}{4}\pi\text{ in}^3$. *Then the spherical scoop is the smallest, followed by the truncated cone scoop, and lastly the cylindrical scoop.*

b. **How many of each scoop would be needed to add a half-cup of sugar to a cupcake mixture? (One-half cup is approximately 7 in^3.) Round your answer to a whole number of scoops.**

The cylindrical scoop is $\frac{1}{4}\pi\text{ in}^3$*, which is approximately* 0.785 in^3*. Let* x *be the number of scoops needed to fill one-half cup.*

$$\begin{aligned} 0.785x &= 7 \\ x &= \frac{7}{0.785} \\ &= 8.9171\ldots \\ &\approx 9 \end{aligned}$$

It would take about 9 *scoops of the cylindrical cup to fill one-half cup.*

The spherical scoop is $\frac{1}{12}\pi\text{ in}^3$*, which is approximately* 0.262 in^3*. Let* x *be the number of scoops needed to fill one-half cup.*

$$\begin{aligned} 0.262x &= 7 \\ x &= \frac{7}{0.262} \\ &= 26.71755\ldots \\ &\approx 27 \end{aligned}$$

It would take about 27 *scoops of the cylindrical cup to fill one-half cup.*

The truncated cone scoop is $\frac{37}{192}\pi\text{ in}^3$*, which is approximately* 0.605 in^3*. Let* x *be the number of scoops needed to fill one-half cup.*

$$\begin{aligned} 0.605x &= 7 \\ x &= \frac{7}{0.605} \\ &= 11.57024\ldots \\ &\approx 12 \end{aligned}$$

It would take about 12 *scoops of the cylindrical cup to fill one-half cup.*

Discussion (15 minutes)

Ask students how they were able to determine the volume of each composite solid in Exercises 1–4. Select a student (or pair) to share their work with the class. Tell them to explain their process using the vocabulary related to the concepts needed to solve the problem. Encourage other students to critique the reasoning of their classmates and to hold them all accountable for the precision of their language. The following questions could be used to highlight MP.1 and MP.2:

- Is it possible to determine the volume of the solid in one step? Explain why or why not.
- What simpler problems were needed in order to determine the answer to the complex problem?
- How did your method of solving differ from the one shown?
- What did you need to do in order to determine the volume of the composite solids?
- What symbols or variables were used in your calculations, and how did you use them?
- What factors might account for minor differences in solutions?
- What expressions were used to represent the figures they model?

Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- As long as no parts of solids overlap, we can add their volumes together.
- We know how to use the formulas for cones, cylinders, spheres, and truncated cones to determine the volume of a composite solid.

> Lesson Summary
>
> **Composite solids are figures comprising more than one solid. Volumes of composites solids can be added as long as no parts of the solids overlap. That is, they touch only at their boundaries.**

Exit Ticket (5 minutes)

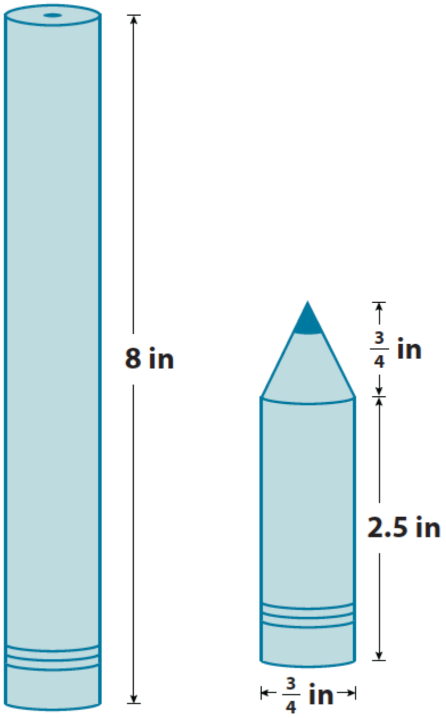
Name ____ Date ________________

Lesson 21: Volume of Composite Solids

Exit Ticket

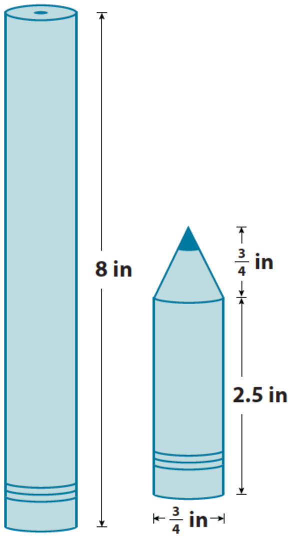
Andrew bought a new pencil like the one shown below on the left. He used the pencil every day in his math class for a week, and now his pencil looks like the one shown below on the right. How much of the pencil, in terms of volume, did he use?

Note: Figures are not drawn to scale.



Exit Ticket Sample Solutions

Andrew bought a new pencil like the one shown below on the left. He used the pencil every day in his math class for a week, and now his pencil looks like the one shown below on the right. How much of the pencil, in terms of volume, did he use?



$$V = \pi(0.375)^2 8$$
$$V = 1.125\pi$$

Volume of the pencil at the beginning of the week was $1.125\pi \text{ in}^3$.

$$V = \pi(0.375)^2 2.5$$
$$V = 0.3515\pi$$

The volume of the cylindrical part of the pencil is $0.3515\pi \text{ in}^3$.

$$V = \frac{1}{3}\pi(0.375)^2(0.75)$$
$$V = \frac{0.1054}{3}\pi$$
$$V = 0.0351\pi$$

The volume of the cone part of the pencil is $0.0351\pi \text{ in}^3$.

$$0.3515\pi + 0.0351\pi = (0.3515 + 0.0351)\pi = 0.3866\pi.$$

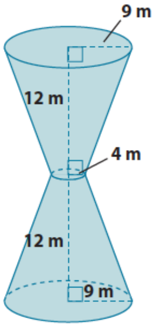
The total volume of the pencil after a week is $0.3866\pi \text{ in}^3$.

$$1.125\pi - 0.3866\pi = (1.125 - 0.3866)\pi = 0.7384\pi.$$

In one week, Andrew used $0.7384\pi \text{ in}^3$ *of the pencil's total volume.*

Problem Set Sample Solutions

1. What volume of sand is required to completely fill up the hourglass shown below? Note: 12 $\mathbf{m}$ is the height of the truncated cone, not the lateral length of the cone.



Let x represent the height of the portion of the cone that has been removed in meters.

$$\begin{aligned}\frac{4}{9} &= \frac{x}{x+12}\\ 4(x+12) &= 9x\\ 4x+48 &= 9x\\ 48 &= 5x\\ \frac{48}{5} &= x\\ 9.6 &= x\end{aligned}$$

The volume of the removed cone is

$$\begin{aligned}V &= \frac{1}{3}\pi 4^2(9.6)\\ &= \frac{153.6}{3}\pi.\end{aligned}$$

The volume of the cone is

$$\begin{aligned}V &= \frac{1}{3}\pi 9^2(21.6)\\ &= \frac{1749.6}{3}\pi.\end{aligned}$$

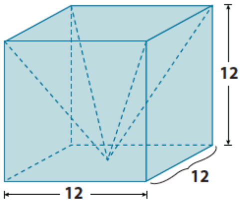
The volume of one truncated cone is

$$\begin{aligned}\frac{1749.6}{3}\pi - \frac{153.6}{3}\pi &= \left(\frac{1749.6}{3} - \frac{153.6}{3}\right)\pi\\ &= \frac{1596}{3}\pi\\ &= 532\ \pi.\end{aligned}$$

The volume of sand needed to fill the hourglass is $1064\pi\ \mathrm{m}^3$.

2.

   a. Write an expression for finding the volume of the prism with the pyramid portion removed. Explain what each part of your expression represents.



$$12^3 - \frac{1}{3}(12^3)$$

The expression 12^3 *is the volume of the cube, and* $\frac{1}{3}(12^3)$ *is the volume of the pyramid. Since the pyramid's volume is being removed from the cube, then we subtract the volume of the pyramid from the cube.*

   b. What is the volume of the prism shown above with the pyramid portion removed?

The volume of the prism is

$$\begin{aligned}V &= 12^3\\ &= 1728.\end{aligned}$$

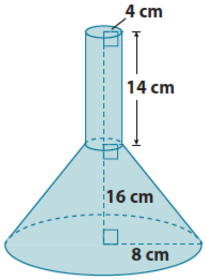
The volume of the pyramid is

$$\begin{aligned}V &= \frac{1}{3}(1728)\\ &= 576.\end{aligned}$$

The volume of the prism with the pyramid removed is 1152 *units*^{3}.

3.

a. **Write an expression for finding the volume of the funnel shown to the right. Explain what each part of your expression represents.**



$$\pi 4^2(14) + \left(\frac{1}{3}\pi 8^2(x+16) - \frac{1}{3}\pi 4^2(x)\right)$$

The expression $\pi 4^2(14)$ *represents the volume of the cylinder. The expression* $\left(\frac{1}{3}\pi 8^2(x+16) - \frac{1}{3}\pi 4^2(x)\right)$ *represents the volume of the truncated cone. The* x *represents the unknown height of the smaller cone. When the volume of the cylinder is added to the volume of the truncated cone, then we will have the volume of the funnel shown.*

b. **Determine the exact volume of the funnel.**

The volume of the cylinder is

$$\begin{aligned} V &= \pi 4^2(14) \\ &= 224\pi. \end{aligned}$$

Let x *be the height of the cone that has been removed in centimeters.*

$$\begin{aligned} \frac{4}{8} &= \frac{x}{x+16} \\ 4(x+16) &= 8x \\ 4x+64 &= 8x \\ 64 &= 4x \\ 16 &= x \end{aligned}$$

The volume of the small cone is

$$\begin{aligned} V &= \frac{1}{3}\pi 4^2(16) \\ &= \frac{256}{3}\pi. \end{aligned}$$

The volume of the large cone is

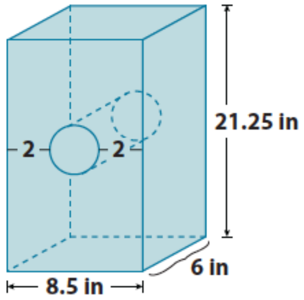
$$\begin{aligned} V &= \frac{1}{3}\pi 8^2(32) \\ &= \frac{2048}{3}\pi. \end{aligned}$$

The volume of the truncated cone is

$$\begin{aligned} \frac{2048}{3}\pi - \frac{256}{3}\pi &= \left(\frac{2048}{3} - \frac{256}{3}\right)\pi \\ &= \frac{1792}{3}\pi. \end{aligned}$$

The volume of the funnel is $224\pi \text{ cm}^3 + \frac{1792}{3}\pi \text{ cm}^3 = 821\frac{1}{3}\pi \text{ cm}^3$.

4. **What is the approximate volume of the rectangular prism with a cylindrical hole shown below? Use 3.14 for π. Round your answer to the tenths place.**



The volume of the prism is

$$V = 8.5(6)(21.25)$$
$$= 1083.75$$

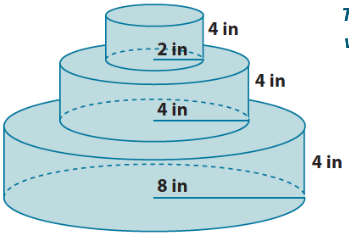
The volume of the cylinder is

$$V = \pi(2.25)^2(6)$$
$$= 30.375\pi$$
$$\approx 95.3775$$

The volume of the prism with the cylindrical hole is

$1083.75\text{ in}^3 - 95.3775\text{ in}^3 = 988.3725\text{ in}^3 \approx 988.4\text{ in}^3$.

5. **A layered cake is being made to celebrate the end of the school year. What is the exact total volume of the cake shown below?**



The bottom layer's volume is

$$V = 8^2\pi(4)$$
$$= 256\pi.$$

The middle layer's volume is

$$V = 4^2\pi(4)$$
$$= 64\pi.$$

The top layer's volume is

$$V = 2^2\pi(4)$$
$$= 16\pi.$$

The total volume of the cake is

$256\pi\text{ in}^3 + 64\pi\text{ in}^3 + 16\pi\text{ in}^3 = (256+64+16)\pi\text{ in}^3 = 336\pi\text{ in}^3$

Lesson 22: Average Rate of Change

Student Outcomes

- Students know how to compute the average rate of change in the height of water level when water is poured into a conical container at a constant rate.

Lesson Notes

MP.1 This lesson focuses on solving one challenging problem that highlights the mathematical practice of making sense of and persevering in solving problems. Working through the problem, students will reach an important conclusion about constant rate and average rate of change. They will learn that given a circumstance where a cone is being filled at a constant rate (the rate at which water is being poured into the cone is constant), the actual rate of change at which the solid is filling up is not constant, hence the “average rate of change.” Throughout the problem, students have to apply many of the concepts learned throughout the year, namely, concepts related to the volume of solids, similarity, constant rate, and rate of change.

The Opening requires a demonstration of the filling of a cone with sand or some other substance.

Classwork

Opening (5 minutes)

Teachers will do a demonstration for students pouring sand (or water, rice, etc.) into an inverted circular cone at a constant rate. Ask students to describe, intuitively, the rate at which the cone is being filled. Specifically, ask students to imagine the cone as two halves, an upper half and a lower half. Which half would fill faster, and why? Teachers can contrast this with a demonstration of water filling a cylinder.

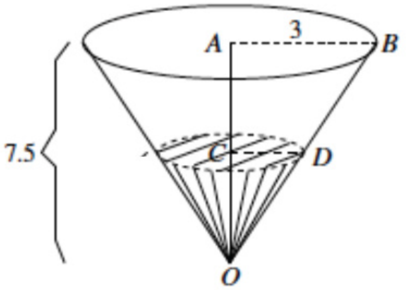
Students should be able to state that the narrower part of the cone is filled more quickly than the wider part of the cone. Therefore, they can conclude that the rate of change of the volume of cone is not constant, and an average rate must be computed. However, the rate of change of the volume of the cylinder is constant because at each increment of the height, the size of the cylinder is exactly the same, which means that the volume increases at a constant rate.

If it is not possible to do a demonstration, you can find a video of a cone being filled at the following location: http://www.youtube.com/watch?v=VEEfHJHMQS8.

Discussion (30 minutes)

Exercise

The height of a container in the shape of a circular cone is 7.5 ft., and the radius of its base is 3 ft., as shown. What is the total volume of the cone?



The volume of the cone is

$$\begin{aligned} V &= \frac{1}{3}\pi 3^2(7.5) \\ &= 22.5\pi. \end{aligned}$$

- If we know the rate at which the cone is being filled, how could we use that information to determine how long it would take to fill the cone?
  - *We could take the total volume and divide it by the rate to determine how long it would take to fill.*
- Water flows into the container (in its inverted position) at a constant rate of $6\ \text{ft}^3$ per minute. Approximately when will the container be filled?

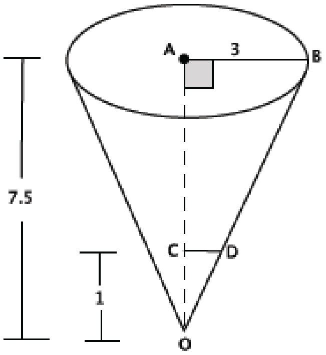
Provide students time to work in pairs on the problem. Have students share their work and their reasoning about the problem.

- *Since the container is being filled at a constant rate, then the volume must be divided by the rate at which it is being filled (using 3.14 as approximation for π and rounding to the hundredths place):*

$$\frac{22.5\pi}{6} \approx 11.78$$

It will take almost 12 minutes to fill the cone at a rate of $6\ \text{ft}^3$ per minute.

- Now, we want to show that even though the water filling the cone flows at a constant rate, the rate of change of the volume in the cone is not constant. For example, if we wanted to know how many minutes it would take for the level in the cone to reach 1 ft., then we would have to first determine the volume of the cone when the height is 1 ft. Do we have enough information to do that?
  - *Yes, we will need to first determine the radius of the cone when the height is* 1 ft.



- What equation can we use to determine the radius when the height is 1 ft.? Explain how your equation represents the situation.
    - *If we let* $|CD|$ *represent the radius of the cone when the height is* 1 ft.*, then*
    $$\frac{3}{|CD|}=\frac{7.5}{1}.$$
    The number 3 *represents the radius of the original cone. The* 7.5 *represents the height of the original cone, and the* 1 *represents the height of the cone for which we are trying to solve.*
- Use your equation to determine the radius of the cone when the height is 1 ft.
    - *The radius when the height is* 1 ft. *is*
    $$\begin{aligned} 3 &= 7.5|CD| \\ \frac{3}{7.5} &= |CD| \\ 0.4 &= |CD|. \end{aligned}$$
- Now, determine the volume of the cone when the height is 1 ft.
    - *Then, we can find the volume of the cone with a height of* 1 ft.:
    $$\begin{aligned} V &= \frac{1}{3}\pi(0.4)^2(1) \\ &= \frac{0.16}{3}\pi. \end{aligned}$$
    Now, we can divide the volume by the rate at which the cone is being filled to determine how many minutes it would take to fill a cone with a height of 1 ft.:
    $$\frac{0.16}{3}\pi \approx 0.167$$
    $$\frac{0.167}{6} \approx 0.028.$$
    It would take about 0.028 *minutes to fill a cone with a height* 1 ft.
- Calculate the number of minutes it would take to fill the cone at 1 ft. intervals. Organize your data in the table below.

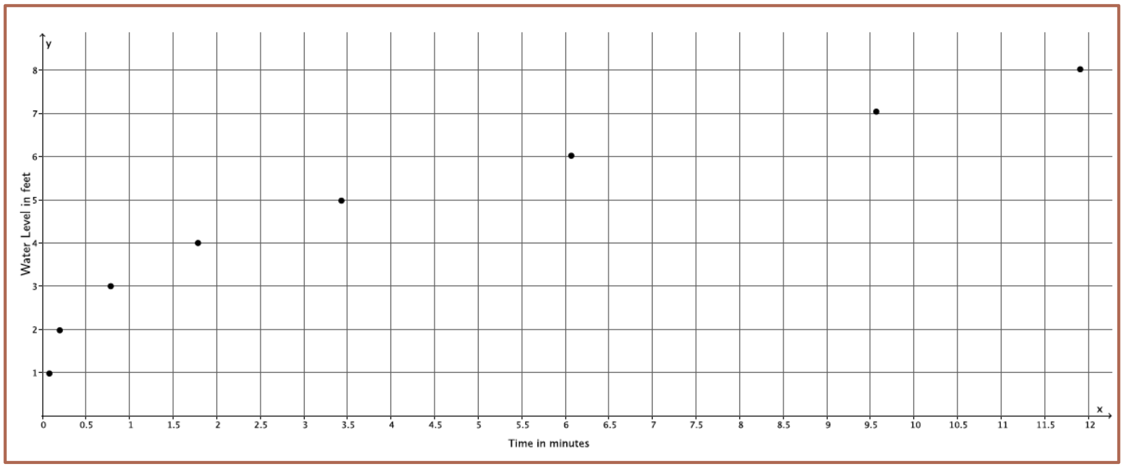
Provide students time to work on completing the table. They should replicate the work above by first finding the radius of the cone at the given heights, then using the radius to determine the volume of the cone, and then determining the time it would take to fill a cone of that volume at the given constant rate. Once most students have finished, continue with the discussion below.

Time (in minutes)	Water Level (in feet)
0.028	1
0.22	2
0.75	3
1.78	4
3.49	5
6.03	6
9.57	7
11.78	7.5

MP.3

- We know that the sand (rice, water, etc.) being poured into the cone is poured at a constant rate, but is the level of the substance in the cone rising at a constant rate? Provide evidence to support your answer.

Provide students time to construct an argument based on the data collected to show that the substance in the cone is not rising at a constant rate. Have students share their reasoning with the class. Students should be able to show that the rate of change (slope) between any two data points is not the same using calculations like $\frac{2-1}{0.22-0.028}=\frac{1}{0.192}=5.2$ and $\frac{7-6}{9.57-6.03}=\frac{1}{3.54}=0.28$ or by graphing the data and showing that it is not linear.



Close the discussion by reminding students of the demonstration at the Opening of the lesson. Ask students if the math supported their conjectures about average rate of change of the water level of the cone.

Closing (5 minutes)

Consider asking students to write a summary of what they learned. Prompt them to include a comparison of how filling a cone is different from filling a cylinder. Another option is to have a whole-class discussion where you ask students how to interpret this information in a real-world context. For example, ask them if they were filling a cylindrical container and a conical container with the same radius and height, which would fill first. Or ask them to discuss whether the rate of change of the volume would be different if we were emptying the cone as opposed to filling it. Would the rate of change in the water level be different if we were emptying the cone as opposed to filling it? How so? What might that look like on a graph?

Summarize, or ask students to summarize, the main points from the lesson:

- We know intuitively that the narrower part of a cone will fill up faster than the wider part of a cone.
- By comparing the time it takes for a cone to be filled to a certain water level, we can determine that the rate of filling the cone is not constant.

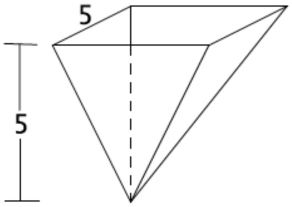
Exit Ticket (5 minutes)

Name ______________________________ Date ______________

Lesson 22: Average Rate of Change

Exit Ticket

A container in the shape of a square base pyramid has a height of 5 ft. and a base length of 5 ft., as shown. Water flows into the container (in its inverted position) at a constant rate of 4 ft^3 per minute. Calculate how many minutes it would take to fill the cone at 1 ft. intervals. Organize your data in the table below.



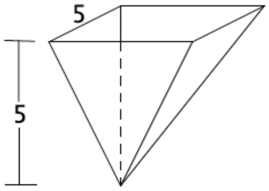
Water Level (in feet)	Area of Base (in feet^2)	Volume (in feet^3)	Time (in minutes)
1			
2			
3			
4			
5			

a. How long will it take to fill up the container?

b. Show that the water level is not rising at a constant rate. Explain.

Exit Ticket Sample Solutions

A container in the shape of a square base pyramid has a height of 5 ft. and a base length of 5 ft., as shown. Water flows into the container (in its inverted position) at a constant rate of 4 ft^3 per minute. Calculate how many minutes it would take to fill the cone at ft. intervals. Organize your data in the table below.



Water Level (in feet)	(Area of Base in feet^2)	Volume (in feet^3)	Time (in minutes)
1	1	$\frac{1}{3}$	0.08
2	4	$\frac{8}{3}$	0.67
3	9	$\frac{27}{3} = 9$	2.25
4	16	$\frac{64}{3}$	5.33
5	25	$\frac{125}{3}$	10.42

a. How long will it take to fill up the container?

It will take approximately 11 min. *to fill up the container.*

b. Show that the water level is not rising at a constant rate. Explain.

$$\frac{2-1}{0.67-0.08} = \frac{1}{0.59} \approx 1.69$$

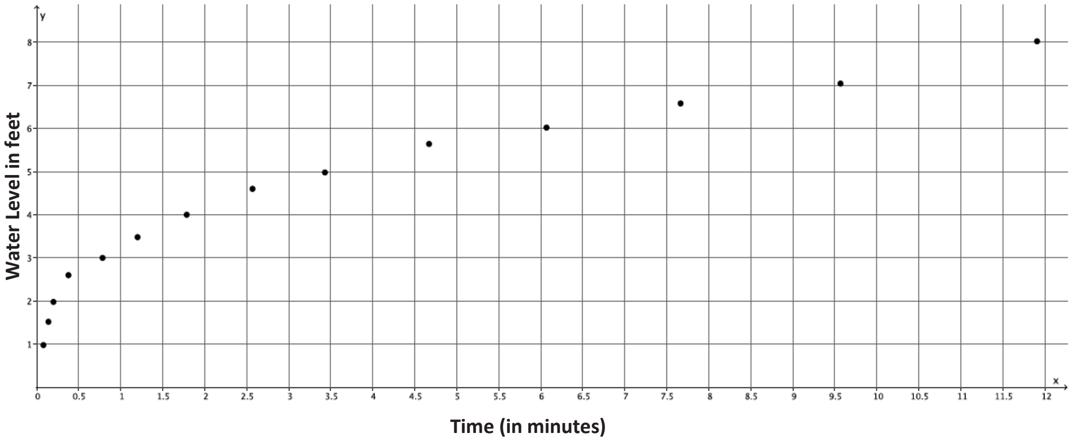
$$\frac{5-4}{10.42-5.33} = \frac{1}{5.09} \approx 0.2$$

The rate at which the water is rising is not the same for the first foot as it is for the last foot. The rate at which the water is rising in the first foot is higher than the rate at which the water is rising in the last foot.

Problem Set Sample Solutions

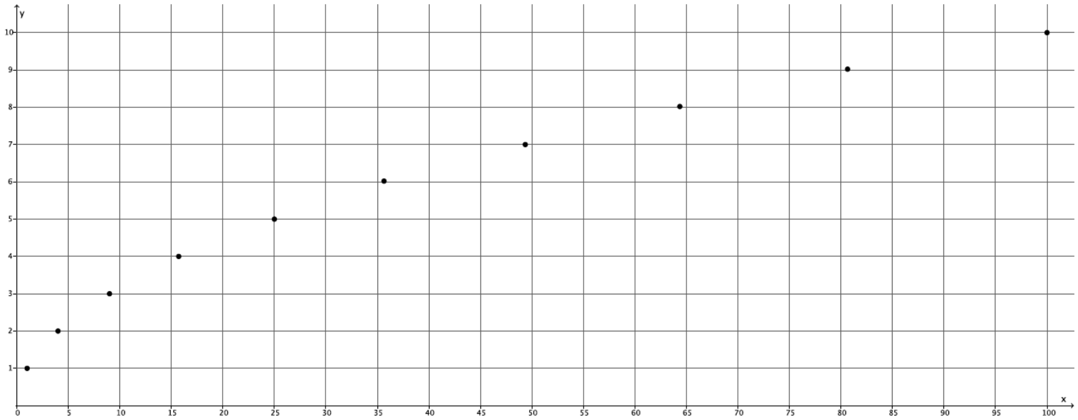
1. **Complete the table below for more intervals of water levels of the cone discussed in class. Then, graph the data on a coordinate plane.**

Time (in minutes)	Water level in feet
0.028	1
0.09	1.5
0.22	2
0.44	2.5
0.75	3
1.2	3.5
1.78	4
2.54	4.5
3.49	5
4.64	5.5
6.03	6
7.67	6.5
9.57	7
11.78	7.5



2. **Complete the table below, and graph the data on a coordinate plane. Compare the graphs from Problems 1 and 2. What do you notice? If you could write a rule to describe the function of the rate of change of the water level of the cone, what might the rule include?**

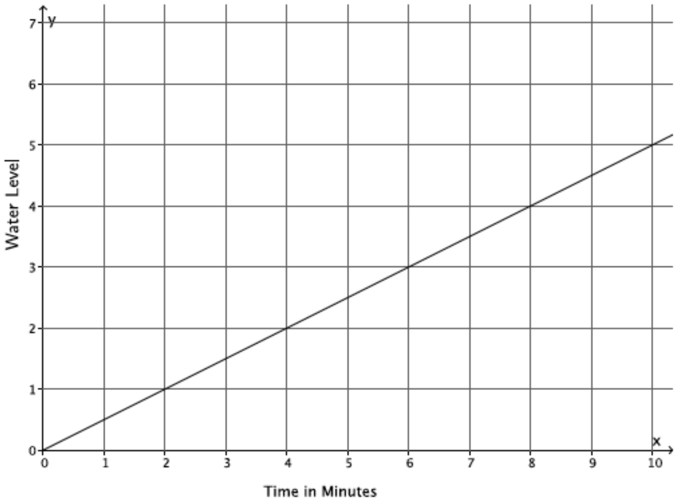
x	$\sqrt{x}$
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
81	9
100	10



The graphs are similar in shape. The rule that describes the function for the rate of change likely includes a square root. Since the graphs of functions are the graphs of certain equations where their inputs and outputs are points on a coordinate plane, it makes sense that the rule producing such a curve would be a graph of some kind of square root.

3. **Describe, intuitively, the rate of change of the water level if the container being filled were a cylinder. Would we get the same results as with the cone? Why or why not? Sketch a graph of what filling the cylinder might look like, and explain how the graph relates to your answer.**

   If the container being filled were a cylinder, we would see a constant rate of change in the water level because there is no narrow or wide part like there is with a cone. Therefore, we would not see the same results as we did with the cone. The rate of change would be the same over any time interval for any given height of the cylinder. The following graph demonstrates this. If a cylinder were being filled at a constant rate, the graph would be linear as shown because the water that would flow into the cylinder would be filling up the same-sized solid throughout.



4. **Describe, intuitively, the rate of change if the container being filled were a sphere. Would we get the same results as with the cone? Why or why not?**

   The rate of change in the water level would not be constant if the container being filled were a sphere. The water level would rise quickly at first, then slow down, then rise quickly again because of the narrower parts of the sphere at the top and the bottom and the wider parts of the sphere around the middle. We would not get the same results as we saw with the cone, but the results would be similar in that the rate of change is nonlinear.

Lesson 23: Nonlinear Motion

Student Outcomes

- Using square roots, students determine the position of the bottom of a ladder as its top slides down a wall at a constant rate.

Lesson Notes

The purpose of this *optional* extension lesson is to incorporate the knowledge obtained throughout the year into a modeling problem about the motion at the bottom of a ladder as it slides down a wall. In this lesson, students will use what they learned about solving multi-step equations from Module 4, which requires knowledge of integer exponents from Module 1. They will also describe the motion of the ladder in terms of a function learned in Module 5 and use what they learned about square roots in this module. Many questions are included to guide students' thinking, but it is recommended that the teacher lead students through the discussion but allow them time to make sense of the problem and persevere in solving it throughout key points within the discussion.

Classwork

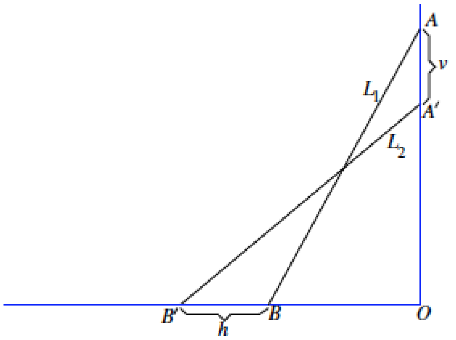
Mathematical Modeling Exercise and Discussion (35 minutes)

There are three phases of the modeling in this lesson: assigning variables, determining the equation, and analyzing results. Many questions are included to guide students' thinking, but the activity may be structured in many different ways, including students working collaboratively in small groups to make sense of and persevere in solving the problem.

Students may benefit from a demonstration of this situation. Consider using a notecard leaning against a box to show what flush means and how the ladder would slide down the wall.

Mathematical Modeling Exercise

A ladder of length L ft. leaning against a wall is sliding down. The ladder starts off being flush with (right up against) the wall. The top of the ladder slides down the vertical wall at a constant speed of v ft. per second. Let the ladder in the position L_1 slide down to position L_2 after 1 second, as shown below.



Will the bottom of the ladder move at a constant rate away from point O?

MP.2

- Identify what each of the symbols in the diagram represent.
    - *O represents the corner where the floor and the wall intersect.*

      L_1 represents the position of the ladder after it has slid down the wall.

      L_2 represents the position of the ladder after it has slid one second after position L_1 down the wall.

      A represents the starting position of the top of the ladder.

      A' represents the position of the top of the ladder after it has slid down the wall for one second.

      v represents the distance that the ladder slid down the wall in one second.

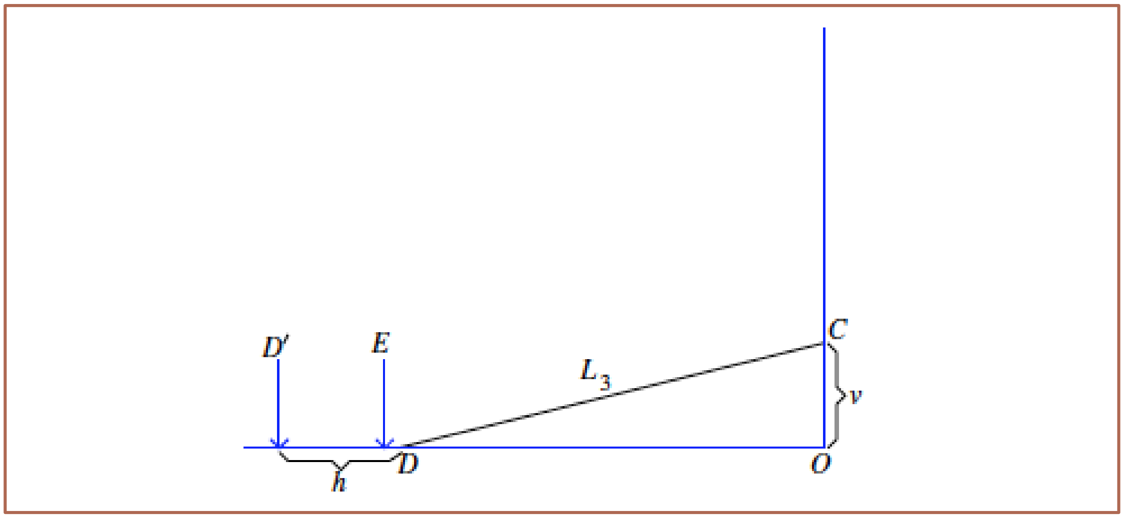
      B represents the starting position of the bottom of the ladder.

      B' represents the position of the bottom of the ladder after it has slid for one second.

      h represents the distance the ladder has moved along the ground after sliding down the wall in one second.
- The distance from point A to point A' is v ft. Explain why.
    - *Since the ladder is sliding down the wall at a constant rate of v ft. per second, then after 1 second, the ladder moves v feet. Since we are given that the time it took for the ladder to go from position L_1 to L_2 is one second, then we know the distance between those points must be v feet.*
- The bottom of the ladder then slides on the floor to the left so that in 1 second it moves from B to B' as shown. Therefore, the average speed of the bottom of the ladder is h ft. per second in this 1-second interval. Will the bottom of the ladder move at a constant rate away from point O?

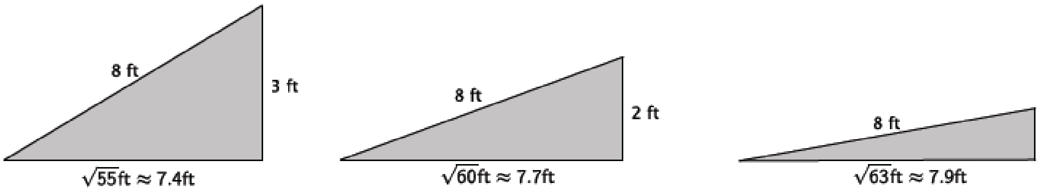
Provide time for students to discuss the answer to the question in pairs or small groups, and then have students share their reasoning. This question is the essential question of the lesson. The answer to this question is the purpose of the entire investigation.

Remind students that functions allow us to make predictions, and then ask students what we would use a function to predict in this situation. Will the function be linear or non-linear? Students should state that we would want the function to predict the location of the bottom of the ladder after sliding down the wall for t seconds. Students should recognize that this situation cannot be described by a linear function. Specifically, if the top of the ladder was v feet from the floor as shown below, it would reach O in one second (because the ladder slides down the wall at a constant rate of v per second). Then after 1 second, the ladder will be flat on the floor, and the foot of the ladder would be at the point where $|EO| = L$, or the length of the ladder.



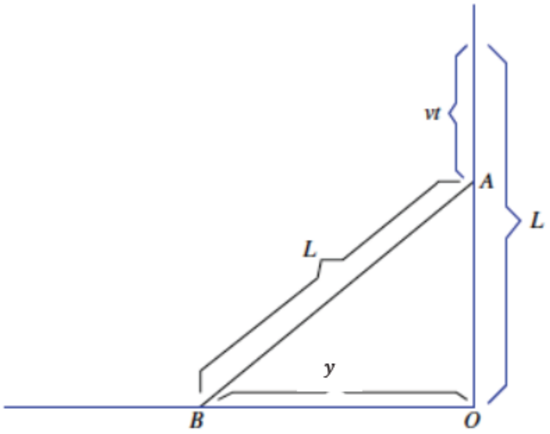
If the rate of change could be described by a linear function, then the point D would move to D' after 1 second, where $|D'D| = h$ ft. (where h is defined as the length the ladder moved from D to D' in one second). But this is impossible. Recall that the length of the ladder is $L = |EO|$. When the ladder is flat on the floor, then at most, the foot of the ladder will be at point E from point O. If the rate of change of the ladder were linear, then the foot of the ladder would be at D' because the linear rate of change would move the ladder a distance of h feet every 1 second. From the picture you can see that $D' \neq E$. Therefore, it is impossible that the rate of change of the ladder could be described by a linear function. Intuitively, if you think about when the top of the ladder, C, is close to the floor (point O), a change in the height of C would produce very little change in the horizontal position of the bottom of the ladder, D. Consider the three right triangles shown below. If we let the length of the ladder be 8 ft., then we can see that a constant change of 1 ft. in the vertical distance, produces very little change in the horizontal distance. Specifically, the change from 3 ft. to 2 ft. produces a horizontal change of approximately 0.3 ft. and the change from 2 ft. to 1 ft. produces a horizontal change of approximately 0.2 ft. A change from 1 ft. to 0 ft., meaning that the ladder is flat on the floor, would produce a horizontal change of just 0.1 ft. (the difference between the length of the ladder, 8 ft. and 7.9 ft.).

Consider the three right triangles shown below, specifically the change in the length of the base as the height decreases in increments of 1 ft.



In particular, when the ladder is flat on the floor so that $C = O$, then the bottom cannot be further left than the point E because $|EO| = L$, the length of the ladder. Therefore, the ladder will never reach point D', and the function that describes the movement of the ladder cannot be linear.

- We want to show that our intuitive sense of the movement of the ladder is accurate. Our goal is to derive a formula, y, for the function of the distance of the bottom of the ladder from O over time t. Because the top of the ladder slides down the wall at a constant rate of v ft. per second, the top of the ladder is now at point A, which is vt ft. below the vertical height of L feet, and the bottom of the ladder is at point B, as shown below. We want to determine the length of $|BO|$, which by definition is the formula for the function, y.



MP.2

- Explain the expression vt. What does it represent?
  - *The expression vt represents the distance the ladder has slid down the wall after t seconds. Since v is the rate at which the ladder slides down the wall, then vt is the distance it slides after t seconds.*
- How can we determine the length of $|BO|$?
  - *The shape formed by the ladder, wall, and floor is a right triangle, so we can use the Pythagorean theorem to find the length of $|BO|$.*
- What is the length of $|AO|$?
  - *The length of $|AO|$ is the length of the ladder L minus the distance the ladder slides down the wall after t seconds (i.e., vt). Therefore, $|AO| = L - vt$.*
- What is the length of the hypotenuse of the right triangle?
  - *The length of the hypotenuse is the length of the ladder, L.*
- Use the Pythagorean theorem to write an expression that gives the length of $|BO|$ (i.e., y).

Provide students time to work in pairs to write the expression for the length of $|BO|$. Give guidance as necessary.

  - *By the Pythagorean theorem*

$$\begin{aligned}(L - vt)^2 + y^2 &= L^2\\ y^2 &= L^2 - (L - vt)^2\\ \sqrt{y^2} &= \sqrt{L^2 - (L - vt)^2}\\ y &= \sqrt{L^2 - (L - vt)^2}\end{aligned}$$

Pause after deriving the equation $y = \sqrt{L^2 - (L - vt)^2}$. Ask students to explain what the equation represents. Students should recognize that the equation gives the distance the foot of the ladder is from the wall, which is $|BO|$.

- By applying the distributive property to $(L - vt)^2$ we get

$$\begin{aligned}(L - vt)^2 &= (L - vt)(L - vt)\\ &= L(L - vt) - vt(L - vt)\\ &= L^2 - Lvt - Lvt + v^2t^2\\ &= L^2 - 2Lvt + v^2t^2\end{aligned}$$

  Then, by substitution, $y = \sqrt{L^2 - (L - vt)^2}$ is equal to

$$\begin{aligned}y &= \sqrt{L^2 - (L^2 - 2Lvt + v^2t^2)}\\ &= \sqrt{L^2 - L^2 + 2Lvt - v^2t^2}\\ &= \sqrt{2Lvt - v^2t^2}\end{aligned}$$

  By the distributive property again,

$$\begin{aligned}y &= \sqrt{2Lvt - v^2t^2}\\ &= \sqrt{vt(2L - vt)}\end{aligned}$$

  At this point we must say something about the possible values of t. For example, what would happen if t were very large? Consider this using some concrete numbers: Suppose the constant rate, v, of the ladder falling down the wall is 2 feet per second, the length of the ladder, L, is 10 feet, and the time t is 100 seconds, what is y equal to?

  - *The value of y is*

$$\begin{aligned} y &= \sqrt{vt(2L - vt)} \\ &= \sqrt{2(100)((2(10) - 2(100))} \\ &= \sqrt{200(20 - 200)} \\ &= \sqrt{200(-180)} \\ &= \sqrt{-36{,}000} \end{aligned}$$

    If the value of t were very large, then the formula would make no sense because the length of $|BO|$ would be equal to the square of a negative number.

- For this reason, we can only consider values of t so that the top of the ladder is still above the floor. Symbolically, $vt \leq L$, where vt is the expression that describes the distance the ladder has moved at a specific rate v for a specific time t. We need that distance to be less than or equal to the length of the ladder.
- What happens when $t = \frac{L}{v}$? Substitute $\frac{L}{v}$ for t in our formula.
  - *Substituting $\frac{L}{v}$ for t,*

$$\begin{aligned} y &= \sqrt{vt(2L - vt)} \\ &= \sqrt{v\left(\frac{L}{v}\right)\left(2L - v\left(\frac{L}{v}\right)\right)} \\ &= \sqrt{L(2L - L)} \\ &= \sqrt{L(L)} \\ &= \sqrt{L^2} \\ &= L \end{aligned}$$

    When $t = \frac{L}{v}$, the top of the ladder will be at the point O and the ladder will be flat on the floor because y represents the length of $|BO|$. If that length is equal to L, then the ladder must be on the floor.

- Back to our original concern: What kind of function describes the rate of change of the movement of the bottom of the ladder on the floor? It should be clear that by the equation $y = \sqrt{vt(2L - vt)}$, which represents $|BO|$ for any time t that the motion (rate of change) is not one of constant speed. Nevertheless, thanks to the concept of a function, we can make predictions about the location of the ladder for any value of t as long as $t \leq \frac{L}{v}$.
- We will use some concrete numbers to compute the average rate of change over different time intervals. Suppose the ladder is 15 feet long, $L = 15$, and the top of the ladder is sliding down the wall at a constant speed of 1 ft. per second, $v = 1$. Then, the horizontal distance of the bottom of the ladder from the wall ($|BO|$) is given by the formula

$$\begin{aligned} y &= \sqrt{1t(2(15) - 1t} \\ &= \sqrt{t(30 - t)} \end{aligned}$$

Determine the outputs the function would give for the specific inputs. Use a calculator to approximate the lengths. Round to the hundredths place.

Input t	Output $y = \sqrt{t(30-t)}$
0	$\sqrt{0} = 0$
1	$\sqrt{29} \approx 5.39$
3	$\sqrt{81} = 9$
4	$\sqrt{104} \approx 10.2$
7	$\sqrt{161} \approx 12.69$
8	$\sqrt{176} \approx 13.27$
14	$\sqrt{224} \approx 14.97$
15	$\sqrt{225} = 15$

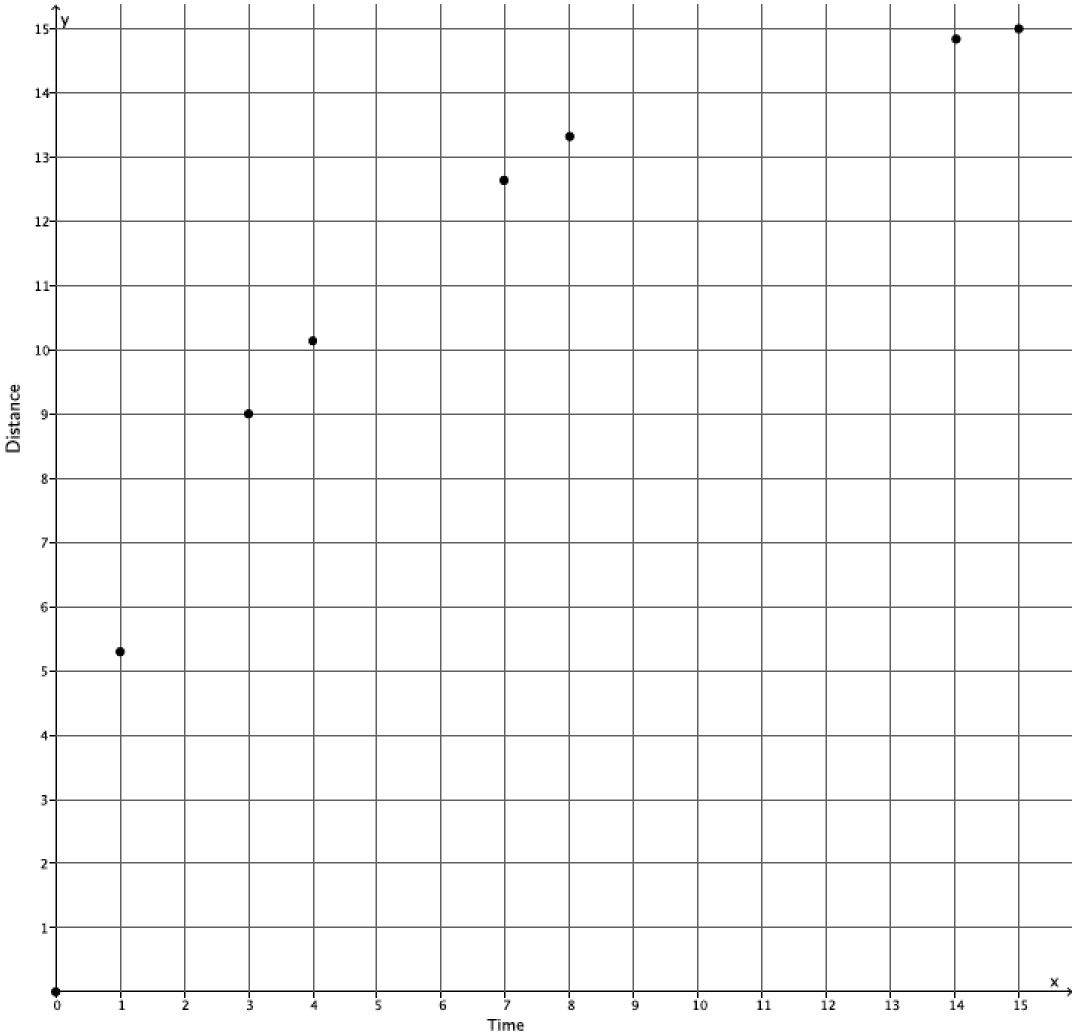
MP.2

- Make at least three observations about what you notice from the data in the table. Justify your observations mathematically with evidence from the table.

Sample observations given below.

  - *The average rate of change between* 0 *and* 1 *second is*
$$\frac{5.39-0}{1-0} = 5.39$$
  - *The average rate of change between* 3 *and* 4 *seconds is*
$$\frac{10.2-9}{4-3} = 1.2$$
  - *The average rate of change between* 7 *and* 8 *seconds is*
$$\frac{13.27-12.69}{8-7} = 0.58$$
  - *The average rate of change between* 14 *and* 15 *seconds is*
$$\frac{15-14.97}{15-14} = 0.03$$

- Now that we have computed the average rate of change over different time intervals, we can make two conclusions: (1) The motion at the bottom of the ladder is not linear, and (2) there is a decrease in the average speeds; that is, the rate of change of the position of the ladder is slowing down as observed in the four 1-second intervals we computed. These conclusions are also supported by the graph of the situation shown on the next page. The data points do not form a line; therefore, the rate of change in position of the bottom of the ladder is not linear.



Closing (5 minutes)

Summarize, or ask students to summarize, the main points from the lesson:

- We know that the motion at the bottom of a ladder as it slides down a wall is not constant because the rate of change of the position of the bottom of the ladder is not constant.
- We have learned how to incorporate various skills to describe the rate of change in the position of the bottom of the ladder and prove that its motion is not constant by computing outputs given by a rule that describes a function, then using that data to show that the average speeds over various time intervals are not equal to the same constant.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 23: Nonlinear Motion

Exit Ticket

Suppose a book is 5.5 inches long and leaning on a shelf. The top of the book is sliding down the shelf at a rate of 0.5 in. per second. Complete the table below. Then, compute the average rate of change in the position of the bottom of the book over the intervals of time from 0 to 1 second and 10 to 11 seconds. How do you interpret these numbers?

Input t	Output $y = \sqrt{0.5t(11 - 0.5t)}$
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	

Exit Ticket Sample Solutions

Suppose a book is 5.5 inches long and leaning on a shelf. The top of the book is sliding down the shelf at a rate of 0.5 in. per second. Complete the table below. Then, compute the average rate of change in the position of the bottom of the book over the intervals of time from 0 to 1 second and 10 to 11 seconds. How do you interpret these numbers?

Input t	Output $y = \sqrt{0.5t(11 - 0.5t)}$
0	$\sqrt{0} = 0$
1	$\sqrt{5.25} \approx 2.29$
2	$\sqrt{10} \approx 3.16$
3	$\sqrt{14.25} \approx 3.77$
4	$\sqrt{18} \approx 4.24$
5	$\sqrt{21.25} \approx 4.61$
6	$\sqrt{24} \approx 4.90$
7	$\sqrt{26.25} \approx 5.12$
8	$\sqrt{28} \approx 5.29$
9	$\sqrt{29.25} \approx 5.41$
10	$\sqrt{30} \approx 5.48$
11	$\sqrt{30.25} = 5.50$

The average rate of change between 0 and 1 seconds is

$$\frac{2.29 - 0}{1 - 0} = \frac{2.29}{1} = 2.29$$

The average rate of change between 10 and 11 seconds is

$$\frac{5.5 - 5.48}{11 - 10} = \frac{0.02}{1} = 0.02$$

The average speeds show that the rate of change of the position of the bottom of the book is not linear. Furthermore, it shows that the rate of change of the bottom of the book is quick at first, 2.29 inches per second in the first second of motion, and then slows down to 0.02 inches per second in the second interval from 10 to 11 seconds.

Problem Set Sample Solutions

1. **Suppose the ladder is 10 feet long, and the top of the ladder is sliding down the wall at a rate of $.8$ ft. per second. Compute the average rate of change in the position of the bottom of the ladder over the intervals of time from 0 to 0.5 seconds, 3 to 3.5 seconds, 7 to 7.5 seconds, 9.5 to 10 seconds, and 12 to 12.5 seconds. How do you interpret these numbers?**

Input t	Output $y = \sqrt{0.8t(20 - 0.8t)}$
0	$\sqrt{0} = 0$
0.5	$\sqrt{7.84} \approx 2.8$
3	$\sqrt{42.24} = 6.5$
3.5	$\sqrt{48.16} \approx 6.94$
7	$\sqrt{80.64} \approx 8.98$
7.5	$\sqrt{84} \approx 9.17$
9.5	$\sqrt{94.24} \approx 9.71$
10	$\sqrt{96} \approx 9.8$
12	$\sqrt{99.84} \approx 9.99$
12.5	$\sqrt{100} = 10$

The average rate of change between 0 and 0.5 seconds is

$$\frac{2.8 - 0}{0.5 - 0} = \frac{2.8}{0.5} = 5.6$$

The average rate of change between 3 and 3.5 seconds is

$$\frac{6.94 - 6.5}{3.5 - 3} = \frac{0.44}{0.5} = 0.88$$

The average rate of change between 7 and 7.5 seconds is

$$\frac{9.17 - 8.98}{7.5 - 5} = \frac{0.19}{0.5} = 0.38$$

The average rate of change between 9.5 and 10 seconds is

$$\frac{9.8 - 9.71}{10 - 9.5} = \frac{0.09}{0.5} = 0.18$$

The average rate of change between 12 and 12.5 seconds is

$$\frac{10 - 9.99}{12.5 - 12} = \frac{0.01}{0.5} = 0.02$$

The average speeds show that the rate of change in the position of the bottom of the ladder is not linear. Furthermore, it shows that the rate of change in the position at the bottom of the ladder is quick at first, 5.6 feet per second in the first half second of motion, and then slows down to 0.02 feet per second in the half second interval from 12 to 12.5 seconds.

2. **Will any length of ladder, L, and any constant speed of sliding of the top of the ladder v ft. per second, ever produce a constant rate of change in the position of the bottom of the ladder? Explain.**

   No, the rate of change in the position at the bottom of the ladder will never be constant. We showed that if the rate were constant, there would be movement in the last second of the ladder sliding down that wall that would place the ladder in an impossible location. That is, if the rate of change were constant, then the bottom of the ladder would be in a location that exceeds the length of the ladder. Also, we determined that the distance that the bottom of the ladder is from the wall over any time period can be found using the formula $y = \sqrt{vt(2L - vt)}$*, which is a non-linear equation. Since graphs of functions are equal to the graph of a certain equation, the graph of the function represented by the formula* $y = \sqrt{vt(2L - vt)}$ *is not a line, and the rate of change in position at the bottom of the ladder is not constant.*

Name ____ Date ____________________

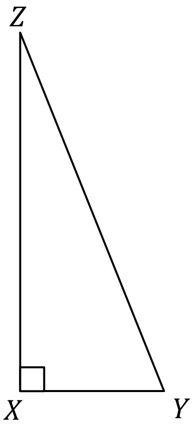
When using a calculator to complete the assessment, use the π key and the full display of the calculator for computations.

1.

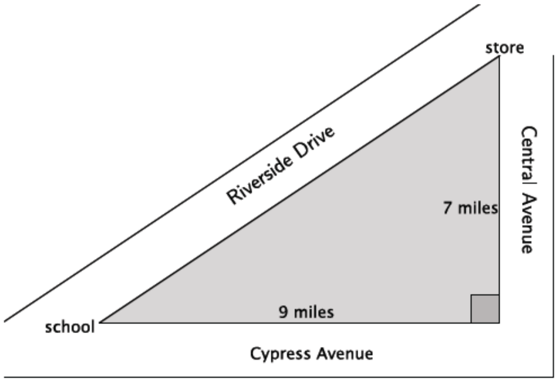
a. Is a triangle with side lengths of 7 cm, 24 cm, and 25 cm a right triangle? Explain.

b. Is a triangle with side lengths of 4 mm, 11 mm, and 15 mm a right triangle? Explain.

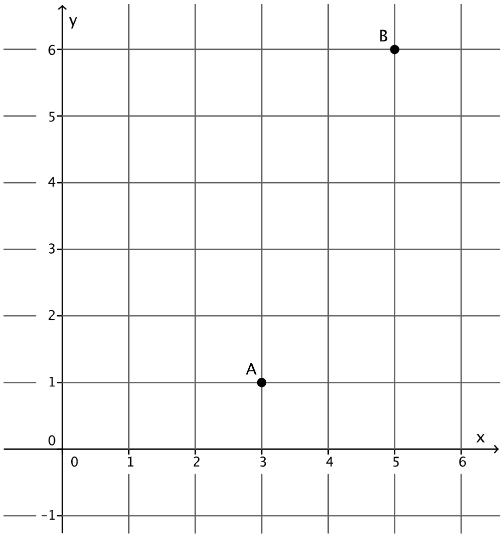
c. The area of the right triangle shown below is 30 ft^2. The segment XY has a length of 5 ft. Find the length of the hypotenuse.



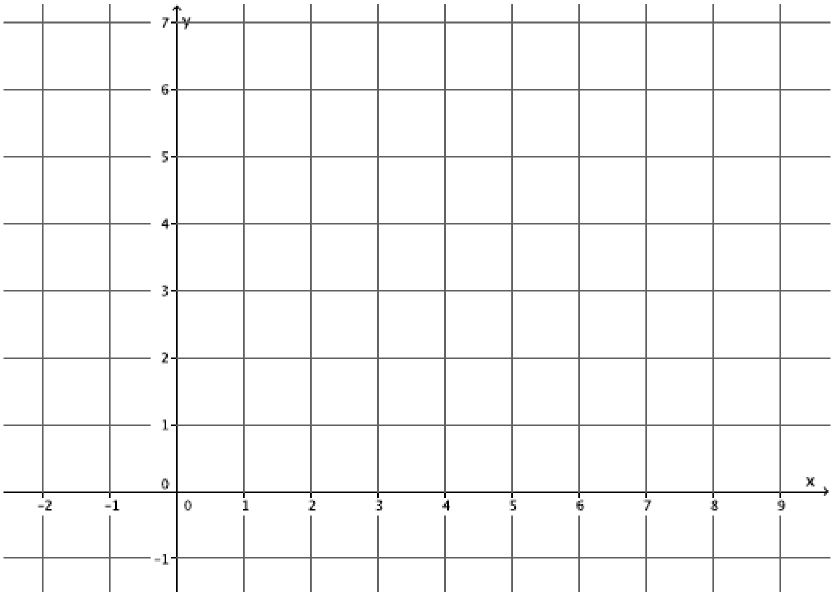
d. Two paths from school to the store are shown below: One uses Riverside Drive, and another uses Cypress and Central Avenues. Which path is shorter? By about how much? Explain how you know.



e. What is the distance between points A and B?



f. Do the segments connecting the coordinates $(-1, 6)$, $(4, 2)$, and $(7, 6)$ form a right triangle? Show work that leads to your answer.



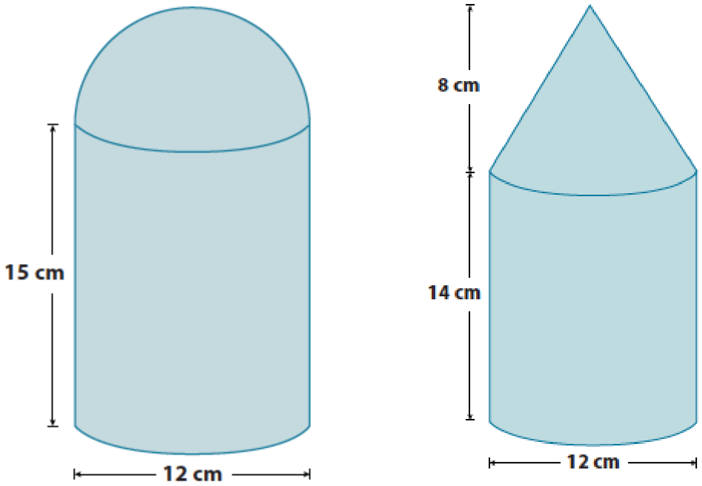
g. Using an example, illustrate and explain the Pythagorean theorem.

h. Using a different example than part (g), illustrate and explain the converse of the Pythagorean theorem.

i. Explain a proof of the Pythagorean theorem and its converse.

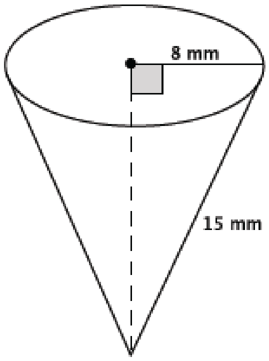
2. Dorothy wants to purchase a container that will hold the most sugar. Assuming each of the containers below can be completely filled with sugar, write a note recommending a container, including justification for your choice.

   Note: Figures not drawn to scale.

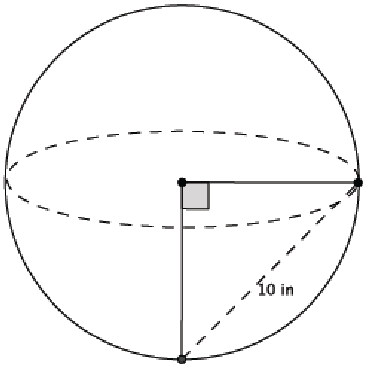


3.

a. Determine the volume of the cone shown below. Give an answer in terms of π and an approximate answer rounded to the tenths place.



b. The distance between the two points on the surface of the sphere shown below is 10 units. Determine the volume of the sphere. Give an answer in terms of π and an approximate answer rounded to a whole number.



c. A sphere has a volume of $457\frac{1}{3}\pi\ \text{in}^3$. What is the radius of the sphere?

A Progression Toward Mastery

Assessment Task Item		**STEP 1** **Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.**	**STEP 2** **Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.**	**STEP 3** **A correct answer with some evidence of reasoning or application of mathematics to solve the problem, OR an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.**	**STEP 4** **A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.**
1	**a–b** **8.G.B.7**	Student does not attempt problem or leaves item blank.	Student correctly responds yes or no to one of parts (a) or (b). Student may or may not provide an explanation. Explanation may show some evidence of mathematical reasoning and references the Pythagorean theorem.	Student correctly responds yes or no to one of parts (a) or (b). Student may make a mathematical error leading to an incorrect response. Student provides an explanation that references the converse of the Pythagorean theorem.	Student correctly responds (a) yes and (b) no. Student may make a mathematical error leading to an incorrect response. Student provides an explanation that references the converse of the Pythagorean theorem.
	c **8.G.B.7**	Student does not attempt problem or leaves item blank.	Student may use the numbers 5 and 30 to determine the length of the hypotenuse. OR Student may calculate the height of the right triangle and names it as the length of the hypotenuse.	Student uses the information in the problem to determine the height of the triangle and the length of the hypotenuse. Student may make a mathematical error leading to an incorrect height and/or an incorrect hypotenuse length.	Student correctly uses the information provided to determine the height of the triangle, 12 ft., and the length of the hypotenuse, 13 ft.

	d **8.G.B.7**	Student does not attempt problem or leaves item blank.	Student may or may not answer correctly. Student is able to calculate one of the paths correctly. <u>OR</u> Student is able to calculate both paths but is unable to approximate the $\sqrt{130}$. Student may or may not provide an explanation. Explanation does not make reference to the Pythagorean theorem.	Student uses the information to calculate the distance of both paths. Student may not approximate $\sqrt{130}$ correctly leading to an incorrect answer. Student may make calculation errors that lead to an incorrect answer. Student's explanation includes the use of the Pythagorean theorem.	Student correctly uses the information provided to calculate that the shortest path is $\sqrt{130} \approx 11.4$ miles. Student's explanation includes the length of the other path as 16 miles. Student's explanation includes the use of the Pythagorean theorem. Student may or may not include an explanation of the approximation of $\sqrt{130}$.
	e **8.G.B.8**	Student does not attempt problem or leaves item blank.	Student does not use the Pythagorean theorem to determine the distance between points A and B. Student may say the distance is 2 units right and 5 units up or another incorrect response.	Student uses the Pythagorean theorem to determine the distance between points A and B but may make a mathematical error leading to an incorrect answer.	Student correctly identifies the length between points A and B as $\sqrt{29}$ units by using the Pythagorean theorem.
	f **8.G.B.8**	Student does not attempt problem or leaves item blank. Student may or may not graph the coordinates. Student finds one of the segment lengths. Student does not make use of the Pythagorean theorem.	Student may or may not answer correctly. Student may make calculation errors in using the Pythagorean theorem. Student finds one or two of the segment lengths but does not compute the third segment length. Student may make calculation errors in using the Pythagorean theorem.	Student answers correctly that the coordinates do not form a right triangle. Student makes use of the Pythagorean theorem to determine all the segment lengths of each segment. Student may make calculation errors in using the Pythagorean theorem.	Student answers correctly that the coordinates do not form a right triangle. Student makes use of the Pythagorean theorem to determine the segment lengths of each segment. Student shows the length of each segment as follows: from $(-1, 6)$ to $(7, 6)$ is 8 units, from $(-1, 6)$ to $(4, 2)$ is $\sqrt{41}$ units, and from $(4, 2)$ to $(7, 6)$ is 5 units.

	g–i **8.G.B.6**	Student does not attempt problem or leaves item blank. Student may use the same example to explain the Pythagorean theorem and its converse. Student's explanation does not demonstrate evidence of mathematical understanding of the Pythagorean theorem or its converse.	Student may or may not use different examples to explain the Pythagorean theorem and its converse. Student may or may not explain a proof of the Pythagorean theorem or its converse. Student's explanation lacks precision and misses many key points in the logic of the proofs. Student's explanation demonstrates some evidence of mathematical understanding of the Pythagorean theorem or its converse.	Student uses different examples to explain the Pythagorean theorem and its converse. Student explains a proof of the Pythagorean theorem and its converse. Student's explanation, though correct, may lack precision or miss a few key points in the logic of the proofs. There is substantial evidence that the student understands the proof of the Pythagorean theorem and its converse.	Student uses different examples to explain the Pythagorean theorem and its converse. Student thoroughly explains a proof of the Pythagorean theorem and its converse. Student uses appropriate mathematical vocabulary and demonstrates with strong evidence understanding of the proofs. Student uses one of the proofs of the Pythagorean theorem found in M2–L15, M3–L13, or M7–L17 and the proof of the converse found in M3–L14 or M7–L18.
2	**8.G.C.9**	Student does not attempt problem or leaves item blank.	Student incorrectly applies the volume formulas leading to incorrect answers. Student may or may not identify the cylinder with the half-sphere on top as the container with the greatest volume. Student may or may not write a note with a recommendation for Dorothy.	Student correctly applies volume formulas but may make a mathematical error leading to an incorrect answer. Student may or may not identify the cylinder with the half-sphere on top as the container with the greatest volume. Student may or may not write a note with a recommendation for Dorothy.	Student correctly calculates the volume of both containers, 684π and 600π. Student correctly identifies the cylinder with the half-sphere on top as the container with the greatest volume. Student writes a note with a recommendation for Dorothy.
3	**a** **8.G.B.7** **8.G.C.9**	Student does not attempt problem or leaves item blank.	Student may or may not determine the height of the cone using the Pythagorean theorem. Student may or may not apply the volume formula for a cone to determine the volume. There is some evidence that the student knows what to do but is unable to apply the correct mathematical concepts to determine the volume.	Student correctly applies the Pythagorean theorem to determine the height of the cone or correctly applies the volume formula for a cone but makes a mathematical error leading to an incorrect answer.	Student correctly calculates the volume of the cone in terms of π as $270.6896542\pi \text{ mm}^3$ and the approximate volume of the cone as 850.4 mm^3.

	b 8.G.B.7 8.G.C.9	Student does not attempt problem or leaves item blank.	Student may or may not determine the radius of the sphere using the Pythagorean theorem. Student may or may not apply the volume formula for a sphere to determine the volume. There is some evidence that the student knows what to do but is unable to apply the correct mathematical concepts to determine the volume.	Student correctly applies the Pythagorean theorem to determine the radius of the sphere or correctly applies the volume formula for a sphere but makes a mathematical error leading to an incorrect answer.	Student correctly calculates the volume of the sphere in terms of π as $471.4045208\pi\ \text{in}^3$ and the approximate volume of the sphere as approximately $1481\ \text{in}^3$.
	c 8.G.C.9	Student does not attempt problem or leaves item blank.	Student uses the formula for the volume of a sphere to write an equation but is unable to solve the equation to determine r. OR Student may use the wrong volume formula leading to an incorrect answer.	Student uses the formula for the volume of a sphere to write an equation. Student may make a mathematical error leading to an incorrect answer. OR Student may leave the answer in the form of $\sqrt[3]{343}$.	Student correctly identifies the radius of the sphere as $7\ \text{in}$.

Name ______________________________ Date ______________

When using a calculator to complete the assessment, use the π key and the full display of the calculator for computations.

1.

a. Is a triangle with side lengths of 7 cm, 24 cm, and 25 cm a right triangle? Explain.

$7^2 + 24^2 = 25^2$

$49 + 576 = 625$

$625 = 625$

Yes. The lengths 7, 24, 25 satisfy the Pythagorean theorem; therefore, it is a right triangle.

b. Is a triangle with side lengths of 4 mm, 11 mm, and 15 mm a right triangle? Explain.

$4^2 \;\; 11^2 = 15^2$

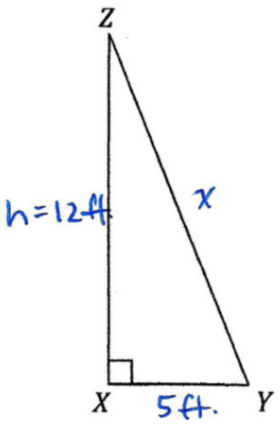
3

$16 + 121 = 225$

$137 \neq 225$

No. The lengths 4, 11, 15 do not satisfy the Pythagorean theorem; therefore, it is not a right triangle.

c. The area of the right triangle shown below is 30 ft^2. The segment XY has a length of 5 ft. Find the length of the hypotenuse.



$\frac{h(5)}{2} = 30$

$5h = 60$

$h = 12$

$5^2 + 12^2 = x^2$

$25 + 144 = x^2$

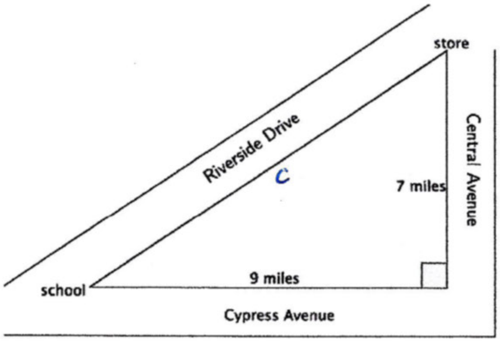
$169 = x^2$

$\sqrt{169} = x$

$13 = x$

The length of the hypotenuse is 13 ft.

d. Two paths from school to the store are shown below: One uses Riverside Drive, and another uses Cypress and Central Avenues. Which path is shorter? By about how much? Explain how you know.

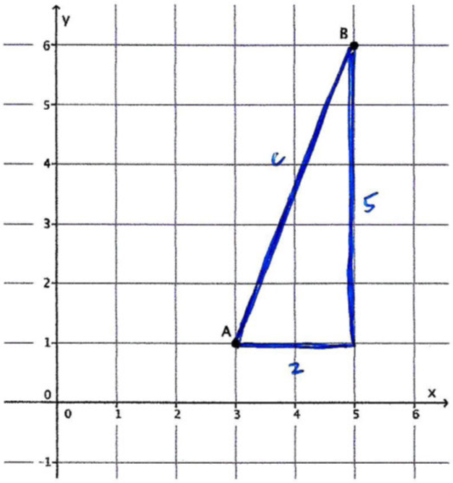


Let c be the hypotenuse in miles.

$$7^2+9^2=c^2$$
$$49+81=c^2$$
$$130=c^2$$
$$\sqrt{130}=\sqrt{c^2}$$
$$\sqrt{130}=c$$
$$11.4\approx c$$

The path along Riverside Drive is shorter, about 11.4 miles, compared to the path along Cypress and Central Avenues, 16 miles. The difference is about 4.6 miles. The Pythagorean theorem allowed me to calculate the distance along Riverside Drive because the three roads form a right triangle.

e. What is the distance between points A and B?

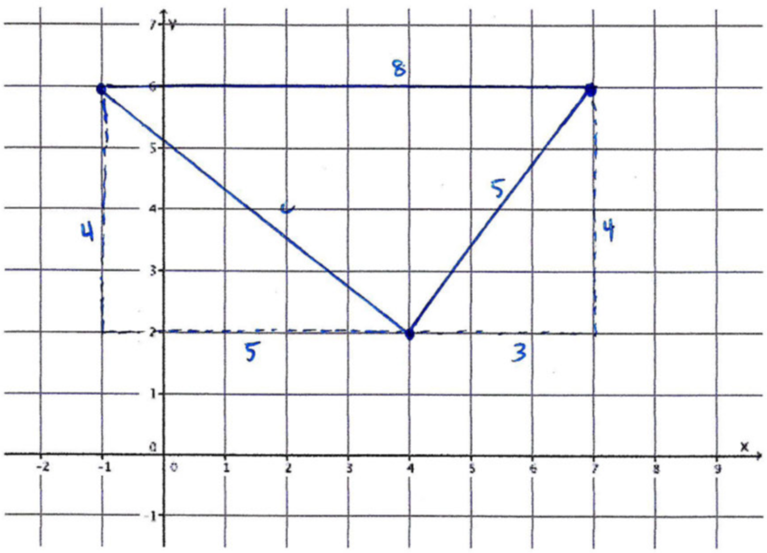


Let c be the difference between points A and B.

$$2^2+5^2=c^2$$
$$4+25=c^2$$
$$29=c^2$$
$$\sqrt{29}=\sqrt{c^2}$$
$$\sqrt{29}=c$$
$$5.4\approx c$$

The distance between points A and B is about 5.4 units.

f. Do the segments connecting the coordinates $(-1, 6)$, $(4, 2)$, and $(7, 6)$ form a right triangle? Show work that leads to your answer.



$3^2 + 4^2 = 5^2$

$9 + 16 = 25$

$25 = 25$

$4^2 + 5^2 = c^2$

$16 + 25 = c^2$

$41 = c^2$

$\sqrt{41} = \sqrt{c^2}$

$\sqrt{41} = c$

$6 < \sqrt{41} < 7$, so side 8 units is the longest.

$5^2 + (\sqrt{41})^2 = 8^2$

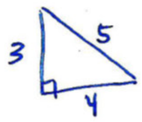
$25 + 41 = 64$

$66 \neq 64$

No. The segments connecting (-1,6), (4,2), and (7,6) do not form a right triangle because their lengths do not satisfy the Pythagorean theorem.

g. Using an example, illustrate and explain the Pythagorean theorem.

Given a right triangle ABC, the sides a, b, c (where c is the hypotenuse) satisfy $a^2 + b^2 = c^2$.



$a = 3, b = 4, c = 5$

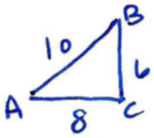
$3^2 + 4^2 = 5^2$

$9 + 16 = 25$

$25 = 25$

h. Using a different example than part (g), illustrate and explain the converse of the Pythagorean theorem.

Given a triangle ABC with side lengths a, b, c (where c is the hypotenuse) that satisfies $a^2+b^2=c^2$, then triangle ABC is a right triangle.



$6^2+8^2=10^2$

$36+64=100$

$100=100$

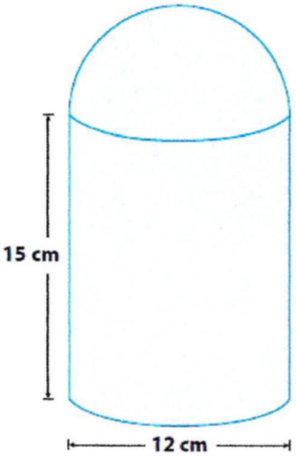
Therefore, triangle ABC is a right triangle because it satisfies the converse of the Pythagorean theorem.

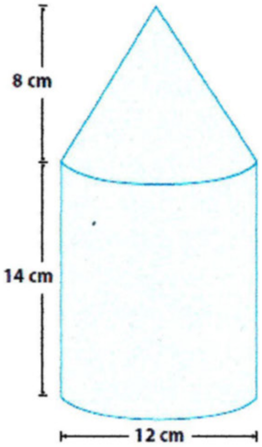
i. Explain a proof of the Pythagorean theorem and its converse.

See rubric to locate proofs of the theorem and its converse within the module.

2. Dorothy wants to purchase a container that will hold the most sugar. Assuming each of the containers below can be completely filled with sugar, write a note recommending a container, including justification for your choice.

Note: Figures not drawn to scale.





Cylinder:

$V = 6^2 \pi (15)$

$= 540\pi$

$\frac{1}{2}$ Sphere:

$V = \frac{1}{2} \left(\frac{4}{3}\right) \pi 6^3$

$= \frac{2}{3}(216)\pi$

$= 144\pi$

Total Volume:

$540\pi + 144\pi = 684\pi$

Cylinder:

$V = 6^2 \pi (14)$

$= 504\pi$

Cone:

$V = \frac{1}{3} \pi (6^2) 8$

$= 96\pi$

Total Volume:

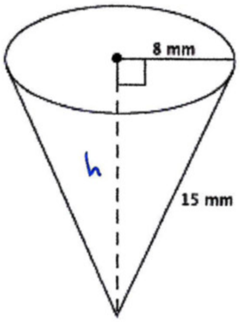
$504\pi + 96\pi = 600\pi$

Dorothy,

You should choose the container with the half sphere on top because it has a greater volume than the container with the cone on top. The containers have volumes of 684π cm^3 and 600π cm^3, respectively. Since 684π is greater than 600π, then the container with the half sphere will hold more sugar compared to the container with the cone on top.

3.

a. Determine the volume of the cone shown below. Give an answer in terms of π and an approximate answer rounded to the tenths place.



$8^2 + h^2 = 15^2$

$64 + h^2 = 225$

$h^2 = 161$

$\sqrt{h^2} = \sqrt{161}$

$h = \sqrt{161}$

$V = \frac{1}{3}\pi(64)(\sqrt{161})$

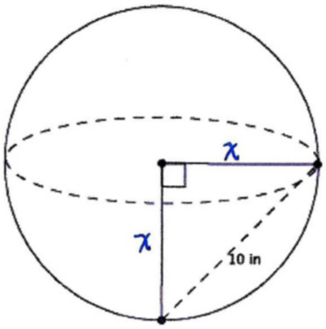
$= \frac{812.0689626}{3}\pi$

$= 270.6896542\pi$

≈ 850.4

The volume of the cone is exactly 270.6896542π mm^3 and approximately 850.4 mm^3.

b. The distance between the two points on the surface of the sphere shown below is 10 units. Determine the volume of the sphere. Give an answer in terms of π and an approximate answer rounded to a whole number.



$x^2 + x^2 = 10^2$

$2x^2 = 100$

$x^2 = 50$

$\sqrt{x^2} = \sqrt{50}$

$x = \sqrt{50}$

$V = \frac{4}{3}\pi(\sqrt{50})^3$

$= \frac{1414.213562}{3}\pi$

$= 471.4045208\pi$

≈ 1481

The volume of the sphere is exactly 471.4045208π in^3 and approximately 1481 in^3.

c. A sphere has a volume of $457\frac{1}{3}\pi$ in^3. What is the radius of the sphere?

$V = 457\frac{1}{3}\pi$

$\frac{\frac{4}{3}\pi r^3}{\pi} = \frac{457\frac{1}{3}\pi}{\pi}$

$\frac{4}{3}r^3 = 457\frac{1}{3}$

$r^3 = 457\frac{1}{3} \times \frac{3}{4}$

$r^3 = \frac{1372}{3} \times \frac{3}{4}$

$r^3 = \frac{1372}{4}$

$r^3 = 343$

$\sqrt[3]{r^3} = \sqrt[3]{343}$

$r = 7$

The radius of the Sphere is 7 in.